On a risk model with dependence between interclaim arrivals and claim sizes

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(Accepted 26 July 2006)

We consider an extension to the classical compound Poisson risk model for which the increments of the aggregate claim amount process are independent. In Albrecher and Teugels (2006), an arbitrary dependence structure among the interclaim time and the subsequent claim size expressed through a copula is considered and they derived asymptotic results for both the finite and infinite-time ruin probabilities. In this paper, we consider a particular dependence structure among the interclaim time and the subsequent claim size and we derive the defective renewal equation satisfied by the expected discounted penalty function. Based on the compound geometric tail representation of the Laplace transform of the time to ruin, we also obtain an explicit expression for this Laplace transform for a large class of claim size distributions. The ruin probability being a special case of the Laplace transform of the time to ruin, explicit expressions are therefore obtained for this particular ruin related quantity. Finally, we measure the impact of the various dependence structures in the risk model on the ruin probability via the comparison of their Lundberg coefficients.

Keywords: Compound Poisson risk model; Gerber-Shiu discounted penalty function; defective renewal equation; Laplace transform of the time to ruin; ruin probability; dependence

1. Introduction

In the actuarial literature, two well-known risk models, namely the classical compound Poisson risk model and the risk model based on a continuous-time renewal process (the latter also referred to as the Sparre Andersen risk model), have been extensively analyzed. Indeed, ruin probabilities and many ruin related quantities such as the marginal and joint defective distributions of the time to ruin, the deficit at ruin, the surplus just prior to ruin and the claim size causing ruin have been extensively studied in these two risk models (see Dufresne and Gerber (1988), Grandell (1991), Dickson (1992, 1993), Gerber and Shiu (1997), Dickson and Hipp (1998), Rolski et al. (1999) and references therein). A unified approach with the discounted penalty function has been proposed by Gerber and Shiu (1998) in the framework of the compound Poisson risk model and has been then studied for some classes of the Sparre Andersen risk model (see Dickson and Hipp (2001), Gerber and Shiu (2005) and Li and Garrido (2004, 2005)).
Note that, for these two risk models, it is explicitly assumed that the interarrival times between two successive claims and the claim amounts are independent. However, there exist many real-world situations for which such an assumption is inappropriate. For instance, for a line of business covering damages due to earthquakes, more considerable damages are expected with a longer period between claims. Albrecher and Boxma (2004) have proposed an extension to the classical compound Poisson risk model in which the distribution of the time between two successive claims depends on the previous claim size. In this paper, we consider an extension to the classical compound Poisson risk model which considers the reverse dependence structure meaning that the distribution of the next claim size depends on the time elapsed since the last claim. We will see that this latter risk model provides a good framework for a ruin theory analysis due to the fact that the independence assumption between the increments of the surplus process is preserved. We mention that an arbitrary dependence structure among the interclaim time and the surplus just prior to ruin are respectively denoted by $\bar{X}_j$ and $W_j$ for all $j \in \mathbb{N}^+$. This allows us, as was done in [Albrecher and Boxma, 2004], to mention the independence assumption between the increments of the surplus process is preserved. We consider the reverse dependence structure meaning that the distribution of the time between two successive claims depends on the previous claim size. Albrecher and Boxma (2004) instance, for a line of business covering damages due to earthquakes, more considerable damages are expected with a longer period between claims. Albrecher and Boxma (2004) where they derived asymptotic results for both the finite and infinite-time ruin probabilities.

In our risk model with time-dependent claim sizes, let $\mathcal{N} = \{N_j, \ t \geq 0\}$ be the claim number process. We assume that $\mathcal{N}$ is a Poisson process with i.i.d. exponential interclaim time r.v.'s $\{W_j, j \in \mathbb{N}^+\}$. For convenience, we denote the claim arrival times $\{T_j, j \in \mathbb{N}^+\}$ by $T_j = W_1 + \ldots + W_j$. The individual claim amount r.v.'s $\{X_j, j \in \mathbb{N}^+\}$ are assumed to be a sequence of strictly positive, i.i.d. r.v.'s with cumulative distribution function (c.d.f.) $F_X(x) = 1 - F_X(x)$ and Laplace transform $\tilde{F}_X$. Contrarily to the classical compound Poisson risk model, we assume that the bivariate random vectors $(W_j, X_j)$ for $j \in \mathbb{N}^+$ are mutually independent but that the r.v.'s $W_j$ and $X_j$ are no longer independent. The total claim amount process $\mathcal{S} = \{S_t, \ t \geq 0\}$ is defined as $S_t = \sum_{j=1}^{N_t} X_j$, where $\Sigma^b$ equals 0 if $b < a$.

Let the surplus process $\mathcal{U} = \{U_t, \ t \geq 0\}$ be defined as $U_t = U_0 + ct - S_t$, where $u (u \in \mathbb{N})$ is the initial surplus level and $c (c \geq 0)$ is the level premium rate. Let $\tau = \inf_{t \geq 0} \{t | U_t < 0\}$ be the time of ruin with $\tau = \infty$ if $U_t \geq 0$ for all $t \geq 0$ (i.e. ruin does not occur). The deficit at ruin and the surplus just prior to ruin are respectively denoted by $|U_\tau|$ and $U_{\tau-}$. The Gerber-Shiu discounted penalty function $m_\delta(u)$ is defined as

$$m_\delta(u) = E[e^{-\delta \psi(U_{\tau-}, |U_\tau|)}1_{\{\tau < \infty\}} | U_0 = u],$$

where $\delta \geq 0$, $w: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is the penalty function and $1_A$ is the indicator of the event $A$. A special case of the Gerber-Shiu discounted penalty function (with $\delta = 0$ and $w(x,y) = 1$ for all $x,y \in \mathbb{R}^+$) is the infinite-time ruin probability $\psi(u) = \Pr(\tau < \infty)$. To ensure that ruin does not occur almost surely, the premium rate $c$ is such that

$$E[c W_j - X_j] > 0,$$

providing a positive safety loading.

In the proposed risk model, it is clear that the increments $\{(X_j - cW_j), j \in \mathbb{N}^+\}$ of the surplus process are still independent. This allows us, as was done in an independent
setting, to obtain with a martingale argument an exponential upper bound for the ruin probability

\[ \psi(u) \leq e^{\rho u}, \]

where \( \rho < 0 \) (if it exists) is the negative solution to \( E[e^{-\rho(X_i - cW_i)}] = 1 \). The quantity \( \rho \) is usually called the Lundberg adjustment coefficient (see e.g. Rolski et al. (1999)). The Lundberg adjustment coefficient is often used as a measure of dangerousness of a surplus process. Moreover, the independence assumption among the increments of the surplus process also implies that the infinite-time ruin probability is a compound geometric tail (as shown later).

The paper is structured as follows: in Section 2, a brief analysis of the aggregate claim amount process is performed in the proposed risk model. In Section 3, we consider a specific dependence structure between the claim amount and the interclaim time r.v.'s. We derive the defective renewal equation for the Gerber-Shiu discounted penalty function in Section 4 while an analysis of some meaningful ruin related quantities with an initial surplus of 0 is performed in Section 5. Then, in Section 6, we consider a family of claim size distributions for which the Laplace transform of the time of ruin has a closed-form expression. Finally, in Section 7, we assess the impact of the dependence structure on the ruin probability via the comparison of Lundberg’s adjustment coefficients.

2. Brief analysis of the aggregate claim amount process

In this section, we derive the first moment of the aggregate claim amount r.v. \( S_t \) as well as providing a formula for its second moment. Based on the order statistics property of the Poisson process, the claim arrival times random vector \((T_1, \ldots, T_n)\) given \( N(t) = n \) has the same distribution as the order statistics of \( n \) independent \((0, t)\) uniformly distributed r.v.'s. Consequently, it follows \( f_{W_1, \ldots, W_n|N(t)=n}(w_1, \ldots, w_n) = \frac{n!}{t^n} \) on \( D = \{(w_1, \ldots, w_n) : 0 \leq w_1 \leq t, \ldots, 0 \leq w_n \leq t - w_{n-1}\} \) which implies

\[
    f_{W_i|N_i=n}(w_i) = \frac{n(t - w_i)^{n-1}}{t^n}, \quad 0 < w_i < t.
\]

Therefore, one can deduce

\[
    E[S_i|N_i = n] = \int_D E[S_i|W = w, N_i = n] f_{W_i|N_i = n}(w) \, dw \\
    = \sum_{i=1}^{n} \int_D E[X_i|W_i = w_i] \frac{n!}{t^n} \, dw \\
    = \sum_{i=1}^{n} \int_0^t E[X_i|W_i = w_i] \frac{n(t - w_i)^{n-1}}{t^n} \, dw_i \\
    = n \int_0^t \mu^{(1)}(w) \frac{n(t - w)^{n-1}}{t^n} \, dw
\]

(2)
where $\mu^{(k)}(w) = E[X_k^k|W_1 = w]$ for $k \in \mathbb{N}$. Finally, (2) leads to
$$E[S_t] = \sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \int_0^t \mu^{(1)}(w) \frac{\lambda}{t} (1 - \frac{w}{t}) e^{-\lambda(t-w)} \left( \sum_{n=0}^{\infty} (\frac{\lambda}{n!})^n e^{-\lambda(t-w)} \right) dw$$
$$= \lambda t \int_0^t \mu^{(1)}(w) \left( 1 - \frac{w}{t} \right) + \frac{1}{t} e^{-\lambda w} dw$$
$$= \lambda t \mu^{(1)}(0) + \int_0^t e^{-\lambda w} \left( 1 - \frac{w}{t} \right) d\mu^{(1)}(w). \quad (3)$$

In the case where the claim size r.v.'s are independent of the time separating two consecutive claims (i.e. $\mu^{(1)}(w) = \mu^{(1)}$ for all $w \geq 0$), (3) becomes $E[S_t] = \lambda t \mu^{(1)}$ which corresponds to the first moment of a compound Poisson r.v. (as expected). Note that, for our dependence structure, Theorem 6.1.12 of Rolski et al. (1999) is verified as well, i.e.
$$\lim_{t \to \infty} \frac{1}{t} E[S_t] = \lambda \lim_{t \to \infty} \int_0^t \mu^{(1)}(w) \left( 1 - \frac{w}{t} \right) e^{-\lambda w} dw + \lim_{t \to \infty} \frac{E[\min(X, t)]}{t}$$
$$= \lambda \int_0^\infty \left( 1 - \frac{w}{t} \right) \mu^{(1)}(w) e^{-\lambda w} 1_{[w \leq t]} dw$$
$$= \lambda \int_0^\infty \mu^{(1)}(w) e^{-\lambda w} dw$$
$$= \lambda E[X_1].$$

using the dominated convergence theorem. A similar procedure leads to an expression for the second moment of the r.v. $S_t$. The result is stated here without proof
$$E[S_t^2] = \int_0^t \mu^{(2)}(w) e^{-\lambda w} (1 + \lambda(t-w)) dw$$
$$+ \int_0^t \int_0^{t-w} \lambda^2 \mu^{(1)}(w) \mu^{(1)}(y) e^{-\lambda w} (2 + 4\lambda(t-w-y) + (\lambda(t-w-y))^2) dy dw. \quad (4)$$

As expected, if $\mu^{(k)}(w) = \mu^{(k)}$ for all $w \geq 0$ and $k = 1,2$, (4) becomes $E[S_t^2] = \lambda t \mu^{(2)} + (\lambda t \mu^{(1)})^2$ which corresponds to the expected total amount of claim if $S_t$ is compound Poisson.

3. Dependence structure

In this section, we consider a dependence structure between the claim amount and the interclaim time r.v.'s $X_k$ and $W_k$ that is mathematically tractable. Namely, we suppose the (conditional) density of $X_k|W_k$ to be defined as a special mixture of two arbitrary density
functions $f_1$ and $f_2$ (with respective means $\mu_1$ and $\mu_2$), i.e.

$$f_{X_k|W}(x) = e^{-\beta W}f_1(x) + (1 - e^{-\beta W})f_2(x), \quad x \geq 0,$$

for $k=1,2,\ldots$. For practical purposes, let $Y_1$ ($Y_2$) be a generic r.v. with density $f_1$ ($f_2$).

From (5), the weight assigned to the c.d.f. $F_1$ is an exponentially decreasing function (at rate $\beta$) of the time elapsed since the last claim $W_k$. The resulting marginal distribution of $X_k$ is

$$f_{X_k}(x) = M_W(-\beta)f_1(x) + (1 - M_W(-\beta))f_2(x) = \frac{\lambda}{\lambda + \beta} f_1(x) + \frac{\beta}{\lambda + \beta} f_2(x),$$

for $k=1,2,\ldots$. For this dependence structure, the positive loading condition (1) is

$$c - \frac{\lambda \mu_1 + \beta \mu_2}{\beta + \lambda} > 0.$$

Note that the covariance between $W$ and $X$ is given by

$$\text{Cov}(W, X) = E[\text{Cov}(X, W|W)] + \text{Cov}(E[X|W, E[W|W])

= \text{Cov}(e^{-\beta W}\mu_1 + (1 - e^{-\beta W})\mu_2, W)

= -\text{Cov}(e^{-\beta W}, W)(\mu_2 - \mu_1)

= \frac{\beta}{(\lambda + \beta)}(\mu_2 - \mu_1),$$

which implies that $\text{Cov}(W, X) > (\prec) 0$ when $E[Y_2] > (\prec) E[Y_1]$. This inequality will be central to the comparison of Lundberg’s adjustment coefficients in the last section of this paper.

Note that this dependence structure can be linked to the one proposed by Albrecher and Boxma (2004). Indeed, let us consider a threshold structure where the threshold r.v.'s \{D_j, j=1,2,\ldots\} are a sequence of i.i.d. exponentially distributed r.v.'s with mean $\frac{1}{\beta}$. We assume the threshold r.v.'s are independent of all the other r.v.'s in the proposed risk model. If the interclaim time $W_j$ is larger (smaller) than the threshold r.v. $D_j$, then the density function of the claim amount $X_j$ is $f_1$ ($f_2$). These assumptions lead to (5) for the density function of $X_k|W_k$.

The risk model with time-dependent claim sizes and dependence structure (5) can be viewed as a more realistic model (than the classical compound Poisson risk model) to approximate the behavior of the aggregate claim process in a natural catastrophe context. Indeed, suppose $W_j$ is the waiting time between the ($j-1$)th and $j$th catastrophes and such an event has two possible intensities, say $I_j=1$ (usual), 2 (severe). It results

$$\Pr(I_j = 1|W_j = w) = e^{-\beta w} = 1 - \Pr(I_j = 2|W_j = w)$$

and hence $\Pr(X_j \leq x|I_j = i) = F_i(x)$ for $i=1,2$. For example, in an earthquake risk context, one can expect that the longer is the time between two events the larger will be the claim amount for the next catastrophe. Hence, more weight should be assigned to the distribution $F_2$ which is chosen with a heavier tail than $F_1$. 

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4. Analysis of the Gerber-Shiu discounted penalty function

In this section, our goal is to show that the Gerber-Shiu discounted penalty function also satisfies a defective renewal equation in the proposed extension to the classical compound Poisson risk model. To identify the form of this defective renewal equation, we first condition $m_d(u)$ on the time and the amount of the first claim leading to

$$m_d(u) = \int_0^\infty \lambda e^{-(\gamma + \beta)t} \int_0^{u+ct} m_d(u + ct - y)(e^{-\beta t} f_1(y) + (1 - e^{-\beta t}) f_2(y)) dy dt$$

$$+ \int_0^\infty \lambda e^{-(\gamma + \beta)t} \int_{u+ct}^\infty w(u + ct, y - (u + ct))(e^{-\beta t} f_1(y) + (1 - e^{-\beta t}) f_2(y)) dy dt$$

$$= \int_0^\infty \tilde{\lambda} e^{-(\gamma + \beta)t} \int_0^t m_d(t - y) \left( e^{-\beta (t-u)} f_1(y) + (1 - e^{-\beta (t-u)}) f_2(y) \right) dy dt$$

$$+ \int_0^\infty \tilde{\lambda} e^{-(\gamma + \beta)t} \int_t^\infty w(t, y - t) \left( e^{-\beta (t-u)} f_1(y) + (1 - e^{-\beta (t-u)}) f_2(y) \right) dy dt$$

$$= \frac{\tilde{\lambda}}{c} \left( \int_0^\infty e^{-(\gamma + \beta + \beta t)} \left( \sigma_{1,\beta}(t) - \sigma_{2,\beta}(t) \right) + \int_0^\infty e^{-(\gamma + \beta + \beta t)} \sigma_{2,\beta}(t) dt \right),$$

(7)

where $\gamma(t) = \int_0^t w(t, y - t)f_1(y) dy$ and

$$\sigma_{i,\beta}(t) = \int_0^t m_d(t - y)f_1(y) dy + \gamma(t),$$

(8)

for $i = 1, 2$. Using the Dickson-Hipp operator $T_r$ defined as

$$T_r f(u) = \int_0^\infty e^{-r(y-u)}f(y) dy,$$

(9)

(see Dickson and Hipp (2001) and Li and Garrido (2004)), (9) can be rewritten as

$$m_d(u) = \frac{\tilde{\lambda}}{c} \left( T_{\gamma + \beta + \beta} \sigma_{1,\beta}(u) - T_{\gamma + \beta + \beta} \sigma_{2,\beta}(u) + T_{\gamma + \beta} \sigma_{2,\beta}(u) \right).$$

(10)

Taking the Laplace transform of (10) combined with the second property of the Dickson-Hipp operator on page 393 of Garrido and Li (2004), it follows

$$\tilde{m}_d(s) = \frac{\tilde{\lambda}}{c} \left( \frac{\tilde{\sigma}_{1,\beta}(\frac{\gamma + \beta + \beta}{c}) - \tilde{\sigma}_{1}(s)}{s - \frac{\gamma + \beta + \beta}{c}} - \frac{\tilde{\sigma}_{2,\beta}(\frac{\gamma + \beta + \beta}{c}) - \tilde{\sigma}_{2}(s)}{s - \frac{\gamma + \beta + \beta}{c}} + \frac{\tilde{\sigma}_{2,\beta}(\frac{\gamma + \beta + \beta}{c}) - \tilde{\sigma}_{2,\beta}(s)}{s - \frac{\gamma + \beta + \beta}{c}} \right)$$

$$= \frac{\tilde{\lambda}}{c} \left( s - \frac{\gamma + \beta + \beta}{c} \right) \left( \frac{\tilde{\sigma}_{1,\beta}(\frac{\gamma + \beta + \beta}{c}) - \tilde{\sigma}_{1}(s)}{s - \frac{\gamma + \beta + \beta}{c}} + \frac{\tilde{\sigma}_{2,\beta}(s)}{s - \frac{\gamma + \beta + \beta}{c}} - \frac{\tilde{\sigma}_{2,\beta}(\frac{\gamma + \beta + \beta}{c})}{s - \frac{\gamma + \beta + \beta}{c}} \right)$$

$$= \tilde{\lambda} \left( \frac{\gamma + \beta + \beta}{c} - s \right) \tilde{\sigma}_{1,\beta}(s) + \frac{\gamma + \beta + \beta}{c} \tilde{\sigma}_{2,\beta}(s) + \tilde{\sigma}_d(s)$$

$$\left( s - \frac{\gamma + \beta + \beta}{c} \right),$$

(11)
where

\[
\tilde{z}_\beta(s) = \frac{\lambda}{c} \left( \left( s - \frac{\lambda + \delta}{c} \right) \left( \tilde{\sigma}_{1,\beta} \left( \frac{\lambda + \delta + \beta}{c} \right) - \tilde{\sigma}_{2,\beta} \left( \frac{\lambda + \delta + \beta}{c} \right) \right) \right)
\]

\[
+ \left( s - \frac{\lambda + \delta + \beta}{c} \right) \tilde{\sigma}_{2,\beta} \left( \frac{\lambda + \delta + \beta}{c} \right).
\]

(12)

Note that, throughout this paper, \( \sim \) is added above a letter to represent the Laplace transform of the corresponding quantity. From (8), we know that \( \tilde{\sigma}_{1,\beta}(s) = \tilde{m}_\beta(s) \tilde{f}_1(s) + \tilde{\gamma}_1(s) \) which means that (11) becomes

\[
\tilde{m}_\beta(s) = \tilde{\mu}_\beta(s) + \tilde{\gamma}_\beta(s)
\]

(13)

Simple modifications to (13) yield

\[
\tilde{m}_\beta(s) = \frac{\tilde{\mu}_\beta(s) + \tilde{\gamma}_\beta(s)}{\tilde{h}_{1,\beta}(s) - \tilde{h}_{2,\beta}(s)},
\]

(14)

where, for \( s \geq 0 \),

\[
\tilde{h}_{1,\beta}(s) = \left( s - \frac{\lambda + \delta + \beta}{c} \right) \left( s - \frac{\lambda}{c} \right),
\]

(15)

\[
\tilde{h}_{2,\beta}(s) = \frac{\lambda}{c} \left( \left( \frac{\lambda + \delta}{c} - s \right) \tilde{f}_1(s) + \frac{\beta}{c} \tilde{f}_2(s) \right),
\]

(16)

and

\[
\tilde{\mu}_\beta(s) = \frac{\lambda}{c} \left( \left( \frac{\lambda + \delta}{c} - s \right) \tilde{\gamma}_1(s) + \frac{\beta}{c} \tilde{\gamma}_2(s) \right).
\]

(17)

Note that the roots of the denominator \( \tilde{h}_{1,\beta}(s) - \tilde{h}_{2,\beta}(s) \) on the right-hand side of (14) are the solutions to the generalized Lundberg equation

\[
E \left[ e^{-\delta W} e^{-\beta X - \delta W} \right] = 1,
\]

(18)

with (5) as the dependence structure between \((X_k, \omega_k)\). Indeed, using (5), the left-hand side of (18) can be written as

\[
E \left[ e^{-\delta W} e^{-\beta X - \delta W} \right] = E \left[ e^{-(\delta - \beta)W} E \left[ e^{-\delta X} | W \right] \right] 
\]

\[
= E \left[ e^{-(\delta - \beta)W} (e^{-\beta W} \tilde{f}_1(s) + (1 - e^{-\beta W}) \tilde{f}_2(s)) \right] 
\]

\[
= \frac{\lambda}{\lambda + \delta + \beta - cs} \tilde{f}_1(s) + \left( \frac{\lambda}{\lambda + \delta - cs} - \frac{\lambda}{\lambda + \delta + \beta - cs} \right) \tilde{f}_2(s) 
\]

\[
= \frac{\lambda(\lambda + \delta - cs) \tilde{f}_1(s) + \lambda \beta \tilde{f}_2(s)}{(\lambda + \delta + \beta - cs)(\lambda + \delta - cs)},
\]
for \( s < \frac{\lambda + \delta}{c} \). Therefore, (18) becomes

\[
\frac{\lambda(\lambda + \delta - cs)\tilde{f}_1(s) + \lambda\beta\tilde{f}_2(s)}{(\lambda + \delta + \beta - cs)(\lambda + \delta - cs)} = 1,
\]

which is equivalent to the equation \( \tilde{h}_{1,\delta}(s) - \tilde{h}_{2,\delta}(s) = 0 \).

Also, it can be shown that, when \( Y_1 \) and \( Y_2 \) are equal in distribution, (14) corresponds to Eq. 2.59 of Gerber and Shiu (1998) which is the Laplace transform of the discounted penalty function in the classical compound Poisson risk model.

To ultimately derive the defective renewal equation for \( m_\delta(u) \), we first examine the denominator on the right-hand side of (14). More precisely, our goal is to find the roots of \( \tilde{h}_{1,\delta}(s) - \tilde{h}_{2,\delta}(s) \) using Rouche’s theorem on a given contour. We consider separately the case where \( \delta > 0 \) and \( \delta = 0 \).

**Proposition 1.** For \( \delta > 0 \), the denominator \( \tilde{h}_{1,\delta}(s) - \tilde{h}_{2,\delta}(s) \) in (14) has exactly 2 roots, say \( s_1(\delta) \) and \( s_2(\delta) \), inside the contour \( C_\delta = \{ s : |z_\delta(s)| = 1 \} \) where \( z_\delta(s) = \frac{\lambda + \beta - cs}{\lambda + \beta} \).

**Proof:** First, we rewrite (15) and (16) in terms of \( z_\delta(s) \)

\[
\tilde{h}_{1,\delta}(s) = \frac{\lambda + \beta}{c} z_\delta(s) \left( \frac{\lambda + \beta}{c} z_\delta(s) - \frac{\beta}{c} \right),
\]

and

\[
\tilde{h}_{2,\delta}(s) = \frac{\lambda}{c} \left( \left( \frac{\lambda + \beta}{c} z_\delta(s) - \frac{\beta}{c} \right) \tilde{f}_1 \left( \frac{\delta}{c} + \frac{\lambda + \beta}{c} (1 - z_\delta(s)) \right) \right.
\]

\[
+ \left. \frac{\beta}{c} \tilde{f}_2 \left( \frac{\delta}{c} + \frac{\lambda + \beta}{c} (1 - z_\delta(s)) \right) \right). \tag{20}
\]

Let us now apply Rouche's theorem to the contour \( C_\delta \). Clearly, \( \tilde{h}_{1,\delta}(s) \) and \( \tilde{h}_{2,\delta}(s) \) are analytic inside \( C_\delta \). To apply Rouche’s theorem, it remains to show that \( |\tilde{h}_{2,\delta}(s)| < |\tilde{h}_{1,\delta}(s)| \) on \( C_\delta \). To do so, we know that \( Re \left( \frac{\lambda + \beta}{c} (1 - z_\delta(s)) \right) \geq 0 \) on \( C_\delta \) which implies

\[
|\tilde{h}_{2,\delta}(s)| \leq \frac{\lambda}{c} \left( \left| \frac{\lambda + \beta}{c} z_\delta(s) - \frac{\beta}{c} \right| \tilde{f}_1 \left( \left| \frac{\delta}{c} + \frac{\lambda + \beta}{c} (1 - z_\delta(s)) \right| \right) \right.
\]

\[
+ \left. \frac{\beta}{c} \tilde{f}_2 \left( \left| \frac{\delta}{c} + \frac{\lambda + \beta}{c} (1 - z_\delta(s)) \right| \right) \right)
\]

\[
\leq \frac{\lambda}{c} \left( \left| \frac{\lambda + \beta}{c} z_\delta(s) - \frac{\beta}{c} \right| \tilde{f}_1 \left( \frac{\delta}{c} + \frac{\lambda + \beta}{c} (1 - z_\delta(s)) \right) \right.
\]

\[
+ \left. \frac{\beta}{c} \tilde{f}_2 \left( \frac{\delta}{c} + \frac{\lambda + \beta}{c} (1 - z_\delta(s)) \right) \right)
\]

\[
\leq \frac{\lambda}{c} \left| \frac{\lambda + \beta}{c} z_\delta(s) - \frac{\beta}{c} \right| + \frac{\beta}{c} \frac{\lambda}{c}. \tag{21}
\]
Given that \( \left| \frac{\lambda + \beta}{c} z_\delta(s) - \frac{\beta}{c} \right| \geq \frac{\lambda + \beta}{c} \left| z_\delta(s) - \frac{\beta}{c} \right| \) on \( C_\delta \), (21) becomes

\[
|\hat{h}_{2,\delta}(s)| < \frac{\lambda + \beta}{c} \left| z_\delta(s) - \frac{\beta}{c} \right| = \frac{\lambda + \beta}{c} \left| z_\delta(s) - \frac{\beta}{c} \right|.
\]  

We also know

\[
|\hat{h}_{1,\delta}(s)| = \left| \frac{\lambda + \beta}{c} z_\delta(s) - \frac{\beta}{c} \right| = \left| \frac{\lambda + \beta}{c} z_\delta(s) - \frac{\beta}{c} \right|.
\]  

Therefore, combining (22) and (23) leads to \(|\hat{h}_{2,\delta}(s)| < |\hat{h}_{1,\delta}(s)|\) on \( C_\delta \). Therefore, one concludes that \( \hat{h}_{1,\delta}(s) \) and \( \hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) \) have the same number of roots inside \( C_\delta \). Since \( \hat{h}_{1,\delta}(s) \) has exactly two zeros inside the unit contour \( C_\delta \), \( \hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) \) must have exactly two roots, say \( s_1(\delta) \) and \( s_2(\delta) \), inside \( C_\delta \). □

We now consider the case \( \delta = 0 \). Its importance is due to various ruin related quantities, such as the ruin probability and the defective joint distribution of the surplus prior to ruin and the deficit at ruin, being special cases of the Gerber-Shiu discounted penalty function with \( \delta = 0 \).

**Proposition 2.** For \( \delta = 0 \), the denominator \( \hat{h}_{1,0}(s) - \hat{h}_{2,0}(s) \) in (14) has exactly 1 root, say \( s_1(0) \), inside the contour \( C_0 = \{ s : |z_0(s)| = 1 \} \) and a second root \( s_2(0) = 0 \) on \( C_0 \).

**Proof:** Let us consider (19) and (20) with \( \delta = 0 \). Clearly, \( \hat{h}_{1,0}(s) \) and \( \hat{h}_{2,0}(s) \) are analytic inside the unit contour \( C_0 \) and continuous on \( C_0 \). Using a similar argument to the one used to prove Proposition 1, one deduces that \( |\hat{h}_{2,0}(s)| < |\hat{h}_{1,0}(s)| \) on \( C_0 \) except at \( s = 0 \) (i.e. \( z_0 = 1 \)) where \( \hat{h}_{1,0}(0) = \hat{h}_{2,0}(0) = \frac{\lambda + \beta}{c} > 0 \). Therefore, the conditions to Rouché’s theorem are not satisfied and we have to rely on an extension to Rouché’s theorem, namely Theorem 1 of Klimenok (2001). Indeed, we have

\[
\frac{d}{dz} \hat{h}_{1,0}(s) \mid_{s=1} - \frac{d}{dz} \hat{h}_{2,0}(s) \mid_{s=1} = \frac{\lambda + \beta}{c} + \left( \frac{\lambda + \beta}{c} \right)^2 - \frac{\lambda + \beta}{c} \left( 1 + \frac{\lambda + \beta}{c} \right) = \frac{\lambda + \beta}{c} + \frac{\lambda + \beta}{c} > 0,
\]

using (6). From Theorem 1 of Klimenok (2001), one concludes that the number of roots of \( \hat{h}_{1,0}(s) - \hat{h}_{2,0}(s) \) inside \( C_0 \) is equal to the number of roots of \( \hat{h}_{1,0}(s) \) inside \( C \) minus 1. Given that \( \hat{h}_{1,0}(s) \) has two roots inside \( C_0 \), \( \hat{h}_{1,0}(s) - \hat{h}_{2,0}(s) \) has one root, say \( s_1(0) \), inside \( C \).

Finally, for \( \delta = 0 \), one deduces from (15) and (16) that a trivial root of \( \hat{h}_{1,0}(s) - \hat{h}_{2,0}(s) \) is \( s_2(0) = 0 \). □

**Remark 3.** Note that, by applying Rouché’s theorem (or its extension) on a different contour and/or by a simple analysis of the functions \( \hat{h}_{1,\delta}(s) \) and \( \hat{h}_{2,\delta}(s) \) (i.e. their first and second derivatives), it can be proven that \( s_1(\delta) \) and \( s_2(\delta) \) are the only two positive and real roots of the denominator \( \hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) \) in (14). Moreover, the two roots \( s_1(\delta) \) and \( s_2(\delta) \) are distinct.
So far, the roots $s_1$ and $s_2$ are written as a function of $\delta$ to show their dependence to the force of interest. For simplicity, we drop this argument from here on. Our objective is now to use the roots to rewrite (14) which will eventually lead to the defective renewal equation for the Gerber-Shiu discounted penalty function. An intermediate result is presented first.

**Lemma 4.** The Laplace transform $\hat{m}_\delta(s)$ of the Gerber-Shiu discounted penalty function satisfies

$$\hat{m}_\delta(s) = \hat{m}_\delta(s)T_1T_2T_1h_{2,\delta}(0) + T_1T_2T_1\beta(0). \quad (24)$$

**Proof:** With $\hat{m}_\delta(s)$ analytic for $\Re s \geq 0$, the roots of the denominator in (14) are also roots of the numerator in (14). Therefore, it follows that $\hat{z}_\delta(s) = -\hat{\beta}_\delta(s_i)$ for $i = 1, 2$. From (12), $\hat{z}(s)$ is a polynomial of degree 1 in $s$. Using the Lagrange interpolating theorem, one deduces

$$\hat{z}_\delta(s) = \hat{z}_\delta(s_1) \frac{(s - s_2)}{(s_1 - s_2)} + \hat{z}_\delta(s_2) \frac{(s - s_1)}{(s_2 - s_1)} = \frac{-\hat{\beta}_\delta(s_1)(s - s_2) - \hat{\beta}_\delta(s_2)(s - s_1)}{s_1 - s_2},$$

which implies

$$\hat{\beta}_\delta(s) + \hat{z}_\delta(s) = \frac{(s - s_2) - (s - s_1)}{s_1 - s_2} \hat{\beta}_\delta(s) - \hat{\beta}_\delta(s_1)(s - s_2) - \hat{\beta}_\delta(s_2)(s - s_1)$$

$$= (s - s_2)\left(\hat{\beta}_\delta(s) - \hat{\beta}_\delta(s_1)\right) - (s - s_1)\left(\hat{\beta}_\delta(s) - \hat{\beta}_\delta(s_2)\right)$$

$$= (s - s_1)(s - s_2)T_1T_2T_1\beta(0) - T_1T_2T_1\beta(0)$$

$$= (s - s_1)(s - s_2)T_1T_2T_1\beta(0). \quad (25)$$

A similar procedure is used to find an alternative expression to the denominator $\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s)$ in (14). From Propositions 1 and 2, we know that $\hat{h}_{1,\delta}(s_i) = \hat{h}_{2,\delta}(s_i)$ for $i = 1, 2$ while, from (15), $\hat{h}_{1,\delta}(s)$ is a polynomial of degree 2 in $s$. Using the Lagrange interpolating theorem, one knows

$$\hat{h}_{1,\delta}(s) = \hat{h}_{1,\delta}(0) \frac{(s - s_1)}{(s_1 - s_2)} + s \left(\hat{h}_{1,\delta}(s_1) \frac{s - s_2}{s_1 - s_2} + \hat{h}_{1,\delta}(s_2) \frac{s - s_1}{s_2 - s_1}\right)$$

$$= \hat{h}_{1,\delta}(0) \frac{(s - s_1)}{(s_1 - s_2)} + s \left(\hat{h}_{2,\delta}(s_1) \frac{s - s_2}{s_1 - s_2} + \hat{h}_{2,\delta}(s_2) \frac{s - s_1}{s_2 - s_1}\right)$$

$$= \hat{h}_{1,\delta}(0) \frac{(s - s_1)}{(s_1 - s_2)} + (s - s_1)(s - s_2) \left(\hat{h}_{2,\delta}(s_1) \frac{1}{s_1 - s_2} + \hat{h}_{2,\delta}(s_2) \frac{1}{s_2 - s_1}\right)$$

$$+ \hat{h}_{2,\delta}(s_1) \frac{s - s_2}{s_1 - s_2} + \hat{h}_{2,\delta}(s_2) \frac{s - s_1}{s_2 - s_1}.$$
\[ h_{1,\delta}(s) - h_{2,\delta}(s) = h_{1,\delta}(0) \frac{(s-s_1)(s-s_2)}{s_1 s_2} \]
\[ + (s-s_1)(s-s_2) \left( \frac{\tilde{h}_{2,\delta}(s_1)}{s_1} - \frac{\tilde{h}_{2,\delta}(s_2)}{s_2} - \frac{1}{s_2 - s_1} \right) \]
\[ - \left( \tilde{h}_{2,\delta}(s) - \tilde{h}_{2,\delta}(s_1) \frac{s-s_2}{s_1-s_2} - \tilde{h}_{2,\delta}(s_2) \frac{s-s_1}{s_2-s_1} \right) \]
\[ = (s-s_1)(s-s_2) T_0 T_{s_2} T_{s_1} h_{1,\delta}(0) - (s-s_1)(s-s_2) \]
\[ \times \left( \frac{\tilde{h}_{2,\delta}(s)}{(s-s_1)(s-s_2)} - \frac{\tilde{h}_{2,\delta}(s_1)}{s_1 - s_2} - \frac{1}{s-s_2} \right) \]
\[ = (s-s_1)(s-s_2) (T_0 T_{s_2} T_{s_1} h_{1,\delta}(0) - T_0 T_{s_2} T_{s_1} h_{2,\delta}(0)). \quad (26) \]

It is easy to prove that \( T_0 T_{s_2} T_{s_1} h_{1,\delta}(0) = 1 \) which implies that (26) becomes
\[ h_{1,\delta}(s) - h_{2,\delta}(s) = (s-s_1)(s-s_2) (1 - T_0 T_{s_2} T_{s_1} h_{2,\delta}(0)). \quad (27) \]

Combining (25) and (27) with (14), one deduces \( \tilde{m}_\delta(s) = \frac{T_0 T_{s_2} T_{s_1} \beta(0)}{1 - T_0 T_{s_2} T_{s_1} \beta(0)} \) which leads to (24). \( \Box \)

Using Lemma 4, we are now in a position to derive the defective renewal equation for \( m_\delta(u) \).

**Theorem 5.** Let
\[ \kappa_\delta = \frac{\lambda}{c} \left( \frac{\lambda + \delta}{c} - s_2 \right) T_0 T_{s_2} T_{s_1} f_1(0) + \frac{\beta}{c} T_0 T_{s_2} T_{s_1} f_2(0) + T_0 T_{s_1} f_1(0) \],
\[ q_{1,\delta} = \frac{\lambda}{\kappa_\delta} \left( \frac{\lambda + \delta}{c} - s_2 \right) T_0 T_{s_2} T_{s_1} f_1(0), \]
\[ q_{2,\delta} = \frac{\beta}{\kappa_\delta} \frac{T_0 T_{s_2} T_{s_1} f_2(0)}{1 - q_{1,\delta} + q_{2,\delta}}. \]

Then, \( 0 \leq q_{1,\delta}, q_{2,\delta} \leq 1 \) with \( 0 \leq q_{1,\delta} + q_{2,\delta} \leq 1 \) and \( m_\delta(u) \) satisfies the following defective renewal equation
\[ m_\delta(u) = \kappa_\delta \int_0^u m_\delta(u-y) g_\delta(y) dy + \zeta_\delta(u), \quad (31) \]

where
\[ g_\delta(y) = q_{1,\delta} \frac{T_{s_2} T_{s_1} f_1(y)}{T_0 T_{s_2} T_{s_1} f_1(0)} + q_{2,\delta} \frac{T_{s_2} T_{s_1} f_2(y)}{T_0 T_{s_2} T_{s_1} f_2(0)} + (1 - q_{1,\delta} - q_{2,\delta}) \frac{T_{s_1} f_1(y)}{T_0 T_{s_1} f_1(0)}. \]
and

\[
\xi_\delta(u) = \frac{\lambda}{c} \left( \frac{\lambda + \delta}{c} - s_2 \right) T_{s_2} T_{s_1} \gamma_1(u) + \frac{\beta}{c} T_{s_2} T_{s_1} \gamma_2(u) + T_{s_1} \gamma_1(u). 
\]

**Proof:** Let us first use (16) and (17) to rewrite (24). From the definition of the Dickson-Hipp operator \( T_1 \), one deduces

\[
T_1 T_{s_2} T_{s_1} h_{2,\delta}(0) = \frac{\lambda}{c} \left( \frac{\lambda + \delta}{c} - s_2 \right) T_{s_2} T_{s_1} f_1(0) + \frac{\beta}{c} T_{s_2} T_{s_1} f_2(0) + \frac{u_f(x) - u_f(x)}{s - s_1} \frac{u_f(x) - u_f(x)}{s - s_1} 
\]

\[
= \frac{\lambda}{c} \left( \frac{\lambda + \delta}{c} - s_2 \right) T_{s_2} T_{s_1} f_1(0) + \frac{\beta}{c} T_{s_2} T_{s_1} f_2(0) 
\]

\[
+ f_1(s) - s_2 T_{s_2} T_{s_1} f_1(0) \frac{f_1(s) - s_1 T_{s_2} T_{s_1} f_1(0)}{s_1 - s_2} 
\]

\[
= \frac{\lambda}{c} \left( \frac{\lambda + \delta}{c} - s_2 \right) T_{s_2} T_{s_1} f_1(0) + \frac{\beta}{c} T_{s_2} T_{s_1} f_2(0) + T_{s_1} f_1(0) 
\]

\[
= \frac{\lambda}{c} \left( \frac{\lambda + \delta}{c} - s_2 \right) T_{s_2} T_{s_1} f_1(0) + \frac{\beta}{c} T_{s_2} T_{s_1} f_2(0) + T_{s_1} f_1(0). 
\]

(32)

By comparing \( h_2 \) and \( \tilde{\beta} \) in respectively (16) and (17), one concludes, using (32), that

\[
T_1 T_{s_2} T_{s_1} h_{2,\delta}(0) = \frac{\lambda}{c} \left( \frac{\lambda + \delta}{c} - s_2 \right) T_{s_2} T_{s_1} \gamma_1(0) + \frac{\beta}{c} T_{s_2} T_{s_1} \gamma_2(0) + T_{s_1} \gamma_1(0) 
\]

\[
= T_1 \xi_\delta(0). 
\]

(33)

Therefore, substituting (32) and (33) in (24), one deduces

\[
\tilde{m}_\delta(s) = \frac{\lambda}{c} \tilde{m}_\delta(s) \left( \frac{\lambda + \delta}{c} - s_2 \right) T_{s_2} T_{s_1} f_1(0) + \frac{\beta}{c} T_{s_2} T_{s_1} f_2(0) + T_{s_1} f_1(0) + T_{s_1} \xi_\delta(0). 
\]

(34)

Inverting the Laplace transform in (34), one finds

\[
m_\delta(u) = \frac{\lambda}{c} \int_0^u m_\delta(u - y) \left( \frac{\lambda + \delta}{c} - s_2 \right) T_{s_2} T_{s_1} f_1(y) + \frac{\beta}{c} T_{s_2} T_{s_1} f_2(y) + T_{s_1} f_1(y) \right) dy + \xi_\delta(u) 
\]

\[
= \frac{\lambda}{c} \int_0^u m_\delta(u - y) \left( \frac{\lambda + \delta}{c} - s_2 \right) T_{s_2} T_{s_1} f_1(y) + \frac{\beta}{c} T_{s_2} T_{s_1} f_2(y) + T_{s_1} f_1(y) \right) dy 
\]

\[
+ \xi_\delta(u). 
\]

(35)
Using (28), (29) and (30), (36) becomes
\[
m_\delta(u) = \kappa_\delta \int_0^u m_y(u-y) \left( q_{1,\delta} \frac{T_{x_0} T_{s_y} f_1(y)}{T_0 T_{s_y} T_{x_0} f_1(0)} + q_{2,\delta} \frac{T_{x_0} T_{s_y} f_2(y)}{T_0 T_{s_y} T_{x_0} f_2(0)} + (1 - q_{1,\delta} - q_{2,\delta}) \frac{T_{s_y} f_1(y)}{T_0 T_{s_y} f_1(0)} \right) dy + \xi_\delta(u),
\]
which corresponds to (31).

For (31) to be a defective renewal equation, it remains to show that \( \kappa_\delta < 1 \). Let us first assume that \( \delta > 0 \). By comparing (33) at \( s = 0 \) to (28), it follows \( \kappa_\delta = T_0 T_{s_y} T_{s_x} h_{2,\delta}(0) \). From (27) at \( s = 0 \),
\[
T_0 T_{s_y} T_{s_x} h_{2,\delta}(0) = 1 - \frac{\tilde{h}_{1,\delta}(0) - \tilde{h}_{2,\delta}(0)}{s_1 s_2} = 1 - \frac{s_1 + \delta + \beta \delta}{c s_2} < 1,
\]
given that \( s_1(\delta) > 0 \) and \( s_2(\delta) > 0 \).

For \( \delta = 0 \), we know, from (28), that
\[
\kappa_0 = \frac{\lambda}{c} \left( \frac{\lambda}{c} T_0 T_{s_y} f_1(0) + \frac{\beta}{c} T_0 T_{s_y} f_2(0) + T_0 T_{s_x} f_1(0) \right),
\]
where
\[
T_0 T_{s_y} f_1(0) = \frac{T_0 T_{s_x} f_1(0)}{s_1} = \frac{\mu_1 - \frac{T_{s_x} f_1(0)}{s_1}}{s_1} = \frac{\mu_1 - \frac{1}{s_1}}{s_1}.
\]
It follows
\[
\kappa_0 = \frac{\lambda}{c} \left( \frac{\mu_1}{s_1} - 1 - \frac{\tilde{f}_1(s_1)}{s_1} \right) + \frac{\beta}{c} \left( \frac{\mu_2}{s_1} - 1 - \frac{\tilde{f}_2(s_1)}{s_1} \right) + \frac{\lambda + \beta}{c} \tilde{f}_1(s_1) + \frac{\beta}{c} \tilde{f}_2(s_1) - s_1 \tilde{f}_1(s_1).
\]
Since \( \tilde{h}_{1,\delta}(s_1) = \tilde{h}_{2,\delta}(s_1) = 0 \), (36) becomes
\[
\kappa_0 = \frac{\lambda}{c} \left( \frac{\mu_1}{s_1} + \frac{\mu_2}{s_1} + \frac{\tilde{f}_1(s_1)}{s_1} - \frac{\tilde{f}_2(s_1)}{s_1} \right) = 1 - \frac{1}{s_1} \left( \frac{\lambda + \beta}{c} - \frac{\lambda}{c} \frac{\mu_1 + \mu_2}{c} \right) < 1,
\]
where the inequality is derived via (6). \( \square \)

5. Analysis when \( u = 0 \)

In this section, we look at some ruin related quantities when \( u = 0 \). Let \( h(x, y, t|0) \) be the joint defective density of the surplus prior to ruin \( (x) \), the deficit at ruin \( (y) \) and the time of ruin \( (t) \) given \( U_0 = 0 \). The discounted joint p.d.f. of the surplus just before ruin and the
deficit at ruin $g_{3,\theta}(x, y|0)$ is given by

$$g_{3,\theta}(x, y|0) = \int_0^\infty e^{-\delta t} h(x, y, t|0) dt.$$  

Let $g_{1,\theta}(x|0) = \int_0^\infty g_{3,\theta}(x, y|0) dy$ be the discounted p.d.f. of the surplus just prior to ruin and $g_{2,\theta}(y|0) = \int_0^\infty g_{3,\theta}(x, y|0) dx$ be the discounted p.d.f. of the deficit at ruin. Our objective is to identify the form of $g_{1,\theta}(x|0)$, $g_{2,\theta}(y|0)$ and $g_{3,\theta}(x, y|0)$ for the proposed risk model.

From the defective renewal equation (31) for $m_\theta(u)$, we know, by letting $u \to 0$, that

$$m_\theta(0) = \frac{\lambda}{c} \left( \left( \frac{\lambda + \delta}{c} - s_2 \right) T_s T_{s_1} \gamma(0) + \frac{\beta}{c} T_s T_{s_1} \gamma_1(0) + T_s \gamma_2(0) \right). \quad (37)$$

We recall that $\gamma_1(u) = \int_u^\infty w(u, y - u)f_1(y)dy$ which implies that (37) can be rewritten as

$$m_\theta(0) = \frac{\lambda}{c} \left( \left( \frac{\lambda + \delta}{c} - s_2 \right) \int_0^\infty \int_0^x e^{-s_2u} e^{-s_1(x-u)} \int_x^\infty w(x, y-x)f_1(y)dydxdu \right)$$

$$= \frac{\lambda}{c} \left( \left( \frac{\lambda + \delta}{c} - s_2 \right) \int_0^\infty \int_0^x e^{-s_2u} e^{-s_1(x-u)} \int_0^x w(x, y)f_1(x+y)dydx \right)$$

$$= \frac{\lambda}{c} \left( \left( \frac{\lambda + \delta}{c} - s_2 \right) \int_0^\infty \int_0^x e^{-s_2u} e^{-s_1(x-u)}f_1(x+y)du \right) dydx$$

$$= \frac{\lambda}{c} \left( \left( \frac{\lambda + \delta}{c} - s_2 \right) \int_0^\infty e^{-s_2u} e^{-s_1(x-u)}f_1(x+y)du \right) dydx.$$  

$$= \frac{\lambda}{c} \left( \left( \frac{\lambda + \delta}{c} - s_2 \right) \int_0^\infty \frac{e^{-s_2x - s_1y}}{s_2 - s_1} \left( \frac{\lambda + \delta}{c} - s_2 \right) f_1(x+y) + \frac{\beta}{c} f_2(x+y) \right) dydx. \quad (38)$$

Using a probabilistic argument of Gerber and Shiu (2005), it is well-known

$$m_\theta(0) = \int_0^\infty \int_0^\infty e^{-\delta t} w(x, y)h(x, y, t|0) dt dy dx = \int_0^\infty \int_0^\infty w(x, y)g_{3,\theta}(x, y|0) dy dx,$$

which, combined with (38), yields

$$g_{3,\theta}(x, y|0) = \frac{\lambda}{c} \left( \frac{e^{-s_2x} - e^{-s_1y}}{s_2 - s_1} \left( \frac{\lambda + \delta}{c} - s_2 \right) f_1(x+y) + \frac{\beta}{c} f_2(x+y) + e^{-s_2x}f_1(x+y) \right). \quad (39)$$
Finally, one can easily find the expression of \( g_{1,\delta}(x) \) and \( g_{2,\delta}(y) \) by combining their respective definitions to (39)

\[
g_{1,\delta}(x)[0] = \frac{\lambda}{c} \left( e^{-s_1 x} - e^{-s_2 x} \right) \left( \frac{\lambda + \delta}{c} - s_2 \right) T_0 f_1(x) + \beta \frac{\lambda}{c} T_0 f_2(x) + e^{-s_1 x} T_0 f_1(x),
\]

and

\[
g_{2,\delta}(y)[0] = \frac{\lambda}{c} \left( \frac{\lambda + \delta}{c} - s_2 \right) T_2 T_s f_1(y) + \beta \frac{\lambda}{c} T_2 T_s f_2(y) + T_s f_1(y).
\]

Of a more practical interest, we can derive respectively the joint distribution of the surplus just prior to ruin and the surplus at ruin, the distribution of the surplus just prior to ruin and the distribution of the surplus at ruin with an initial surplus 0

\[
g_{3,\delta}(x, y)[0] = \frac{\lambda}{c} \left( \frac{\lambda - e^{-s_1 x}}{s_1} \left( \frac{\lambda + \delta}{c} f_1(x + y) + \beta \frac{\lambda}{c} f_2(x + y) \right) + e^{-s_1 x} f_1(x + y) \right),
\]

\[
g_{1,0}(x)[0] = \frac{\lambda}{c} \left( \frac{1 - e^{-s_1 x}}{s_1} \left( \frac{\lambda}{c} T_0 f_1(x) + \beta \frac{\lambda}{c} T_0 f_2(x) \right) + e^{-s_1 x} T_0 f_1(x) \right),
\]

and

\[
g_{2,0}(y)[0] = \frac{\lambda}{c} \left( \frac{\lambda}{c} T_s T_0 f_1(y) + \beta \frac{\lambda}{c} T_s T_0 f_2(y) + T_s f_1(y) \right),
\]

simply by replacing \( s_2 \) by 0 and \( \delta \) by 0 in (39), (40) and (41) respectively. For the proposed risk model, the ladder height distribution of the corresponding random walk is given by

\[
\frac{g_{2,0}(y)[0]}{\psi(0)} = \frac{\xi T_s T_0 f_1(y)}{\xi T_s T_0 f_1(y) + \xi T_0 T_s f_2(y) + T_0 T_s f_1(y)}.
\]

6. The Laplace transform of the time of ruin and the ruin probability

In this section, we focus on the Gerber-Shiu discounted penalty function with \( w(x, y) = 1 \) for all \( x, y \geq 0 \). Let \( v_\delta(u) = E[e^{-\delta \tau}|\tau < \infty] \) be the Laplace transform of the time to ruin. For \( w(x, y) = 1 \) for all \( x, y \geq 0 \), one deduces that

\[
\frac{\xi T_s T_0 T_s T_0 h_2(0)}{1 - T_s T_0 T_s T_0 h_2(0)} = \frac{1}{\kappa^2} \left( \frac{\xi T_s T_s T_0 T_0 h_2(0) - T_s T_s T_0 T_0 h_2(0)}{1 - T_s T_s T_0 T_0 h_2(0)} \right).
\]

Using (28), (42) can be rewritten as

\[
\frac{\xi T_s T_0 T_s T_0 h_2(0)}{1 - T_s T_0 T_s T_0 h_2(0)} = \frac{1}{\kappa^2} \left( \frac{1 - \kappa}{1 - \kappa \delta T_s T_0 T_s T_0 h_2(0)} \right).
\]
which implies that \( \{v_\delta(u), u \geq 0\} \) is a compound geometric tail.

To find an explicit expression for \( v_\delta(u) \) with Laplace transform (42), we assume that the claim size density functions \( f_1 \) and \( f_2 \) to be Coxian distributed of order \( n_1 \) and \( n_2 \) \((n_1, n_2 \) being positive integers), i.e.

\[
\tilde{f}_j(s) = \frac{\lambda_j^r + s\beta_j(s)}{\prod_{k=1}^{n_j}(s + \lambda_{j,k})}, \quad s \in \mathbb{C}
\]

where \( \beta_j(s) \) is a polynomial of degrees \( n_j - 2 \) (or less) and \( \lambda_j^r = \prod_{k=1}^{n_j} \lambda_{j,k} \). Note that the results of this section also hold in the more general case where both the Laplace transforms \( f_1(s) \) and \( f_2(s) \) are rationally distributed.

Under (44), (14) can be rewritten as

\[
\tilde{v}_\delta(s) = \frac{\tilde{\lambda}(\frac{\lambda + \delta}{c} - s)T_0 f_1(0) + \frac{\lambda}{c}T_0 f_2(0) + \tilde{z}_\delta(s)}{(s - \frac{\lambda + \delta + \beta}{c})(s - \frac{\lambda + \beta}{c}) - \frac{\lambda}{c} \left( \prod_{k=1}^{n_1}(s + \lambda_{1,k}) \right) \left( \prod_{k=1}^{n_2}(s + \lambda_{2,k}) \right)}
\]

\[
= \frac{\tilde{\lambda}(s - I_{1,\delta}(s))}{s(I_{1,\delta}(s) - I_{2,\delta}(s))}, \quad (45)
\]

where

\[
\tilde{\lambda}(s) = \left( \prod_{k=1}^{n_1}(s + \lambda_{1,k}) \right) \left( \prod_{k=1}^{n_2}(s + \lambda_{2,k}) \right) \left( \frac{\lambda}{c} \left( \frac{\lambda + \delta}{c} - s \right) f_1(0) + \frac{\beta}{c} f_2(0) \right) + s\tilde{z}_\delta(s),
\]

\[
I_{1,\delta}(s) = \left( s - \frac{\lambda + \delta}{c} \right) \left( s - \frac{\lambda + \beta}{c} \right) \left( \prod_{k=1}^{n_1}(s + \lambda_{1,k}) \right) \left( \prod_{k=1}^{n_2}(s + \lambda_{2,k}) \right), \quad (46)
\]

and

\[
I_{2,\delta}(s) = \frac{\lambda}{c} \left( \frac{\lambda + \delta}{c} - s \right) \left( \prod_{k=1}^{n_1}(s + \lambda_{1,k}) \right) \left( \prod_{k=1}^{n_2}(s + \lambda_{2,k}) \right)
\]

\[
+ \frac{\beta}{c} \left( \prod_{k=1}^{n_1}(s + \lambda_{1,k}) \right).
\]

By comparing (42) and (45), one deduces \( \tilde{\lambda}(s) = \kappa_\delta I_{1,\delta}(s) \) leading to

\[
s\tilde{v}_\delta(s) = \frac{\kappa_\delta I_{1,\delta}(s) - I_{2,\delta}(s)}{I_{1,\delta}(s) - I_{2,\delta}(s)}. \quad (48)
\]

To ultimately derive a closed-form expression for \( \{v_\delta(u), u \geq 0\} \), we first examine the denominator on the right-hand side of (48). More precisely, our goal is to find the roots of \( I_{1,\delta}(s) - I_{2,\delta}(s) \) on the left-half complex plane using Rouche’s theorem. To do so, let the contour \( D_r \) consists of the imaginary axis from \( -ir \) to \( +ir \) and a semi-circle in the left halfplane with radius \( r \) and origin \( O \). Let \( D \) be the limiting contour, i.e. \( D_r \) with \( r \to \infty \).

**Theorem 6.** The denominator \( I_{1,\delta}(s) - I_{2,\delta}(s) \) in (48) has exactly \( n_1 + n_2 \) roots, say \( R_i(\delta) \) for \( i = 1, \ldots, n_1 + n_2 \), inside the contour \( D \).
We point out that $\tilde{I}_{1,\delta}(s) - \tilde{I}_{2,\delta}(s)$ has also 2 roots, $s_1$ and $s_2$, with $s_1, s_2 \geq 0$ (Propositions 1 and 2).

**Theorem 7.** Assuming the roots \( \{-R_i(\delta)\}, i = 1, \ldots, n_1+n_2 \} \) distinct, a closed-form expression for \( \{v_{\delta}(u), u \geq 0\} \) is given by

\[
    v_{\delta}(u) = \sum_{i=1}^{n_1+n_2} \xi_i e^{-R_i u},
\]

where

\[
    \xi_i = \frac{\kappa_1 \tilde{I}_{1,\delta}(-R_i) - \tilde{I}_{2,\delta}(-R_i)}{\tilde{I}_{2,\delta}(0) - \tilde{I}_{1,\delta}(0)} \frac{1}{s + R_i} \prod_{k=1, k \neq i}^{n_1+n_2} \frac{R_k}{R_i}.
\]

**Proof:** Using the Lagrange interpolating polynomial on the denominator and the numerator in (48), one finds

\[
    \tilde{I}_{1,\delta}(s) - \tilde{I}_{2,\delta}(s) = (\tilde{I}_{1,\delta}(0) - \tilde{I}_{2,\delta}(0)) \frac{s - s_1}{s_1} \frac{s - s_2}{s_2} \prod_{i=1}^{n_1+n_2} \frac{s + R_i}{R_i},
\]

and, using (43) at $s = 0$ and $\lim_{s \to 0} s \tilde{v}_{\delta}(s) = 0$,

\[
    \kappa_1 \tilde{I}_{1,\delta}(s) - \tilde{I}_{2,\delta}(s) = \sum_{i=1}^{n_1+n_2} (\kappa_1 \tilde{I}_{1,\delta}(-R_i) - \tilde{I}_{2,\delta}(-R_i)) \frac{s - s_1}{-R_i} \frac{s - s_2}{s_2} \prod_{k=1, k \neq i}^{n_1+n_2} \frac{s + R_k}{R_k - R_i}.
\]

Combining (50) and (51) to (48), one concludes

\[
    \tilde{v}_{\delta}(s) = \sum_{i=1}^{n_1+n_2} \xi_i \frac{1}{s + R_i}.
\]

The inversion of the Laplace transform in (52) yields (49).
7. Numerical illustration and impact of the dependence structure

First, we start with a numerical example. We compare 3 risk models with identical marginal distributions for the r.v.’s $X$ and $W$ but different dependence structures:

- Model A: Claim-dependent risk model with $Y_1 \sim \text{Exp}(2.5)$, $Y_2 \sim \text{Exp}(0.5)$ and $\beta = \frac{1}{3}$.
- Model B: Claim-dependent risk model with $Y_1 \sim \text{Exp}(0.5)$, $Y_2 \sim \text{Exp}(2.5)$ and $\beta = 3$.
- Model IND: classical compound Poisson risk model; $(X, cW)$ are independent.

For the three models, $W \sim \text{Exp}(1)$, the marginal of $X$ is a mixture of two exponentials

$$f_X(x) = \frac{3}{4}(1 - e^{-2.5x}) + \frac{1}{4}(1 - e^{-0.5x}),$$

and $c = 1$. From (49), we can find

$$\psi_A(u) = 0.69047e^{-0.16667u} + 0.08471e^{-1.68614u},$$

and

$$\psi_B(u) = 0.78455e^{-0.11092u} + 0.02669e^{-2.12729u}.$$ 

In the model IND, we know, from e.g. Panjer and Willmot (1992), that

$$\psi_{IND}(u) = 0.74640e^{-0.13398u} + 0.05360e^{-1.86605u}.$$ 

It is interesting to mention that the form of $\psi_A(u)$ and $\psi_B(u)$ is the same as the one of $\psi_{IND}(u)$.

In figure 1, we observe $\psi_A(u) \leq \psi_{IND}(u) \leq \psi_B(u)$ for $u \geq 0$. The ordering of the ruin probabilities is not proved yet. In these circumstances, some authors order the adjustment coefficients attached to models A, B and IND. In particular, we want to verify if introducing a dependence relation among the interarrival times and the claim sizes leads to an increment or a decrement in the value of the adjustment coefficients, commonly used in the exponential upper bound ($e^{ru}$) for the ruin probability. Albrecher and Teugels (2006) consider a similar problem in a more general setting. Similar studies regarding the impact of assumed structures of dependence in risk processes on the Lundberg adjustment coefficient are also made in several papers (see e.g. Cossette and Marceau (2000), Müller and Pflug (2001), Denuit et al. (2002), Juri (2002), Macci et al. (2005)).

Let $\rho$ and $\rho_{IND}$ be respectively the negative solutions (if they exist) to

$$E[e^{-\rho(X - cW)}] = 1,$$

and

$$E[e^{-\rho_{IND}(X - cW_{IND})}] = E[e^{-\rho X_{IND}}]E[e^{cW_{IND}}] = 1.$$

Our goal is to qualify the ordering of $\rho$ and $\rho_{IND}$ under some conditions.

For that purpose, we recall some results on stochastic ordering. First, a r.v. $Z_1$ is said to be stochastically increasing (decreasing) in $Z_2$ if $\Pr(Z_1 > z_1 | Z_2 = z_2)$ is increasing (decreasing) with $z_2$ for all $z_1$. If $Z_1$ is stochastically increasing (decreasing) in $Z_2$, then
\[(Z_1, Z_2)\] are positively (negatively) associated for which we have

\[E[g_1(X_1)g_2(X_2)] \geq (\leq) E[g_1(X_1)]E[g_2(X_2)]\]

for all real-valued functions \(g_1\) and \(g_2\) which are increasing (in both components) and are such that their expectations exist (for more details, see Joe (1997) and Joag-Dev and Proschan (1983)).

From (5) with \(F_{Y_1}(x) \geq (\leq) F_{Y_2}(x)\) for all \(x \geq 0\), one concludes that \(X\) is stochastically increasing (decreasing) in \(W\). It implies that \((X, W)\) are positively (negatively) associated or, equivalently, that \((-X, cW)\) are negatively (positively) associated. Since the exponential function is increasing and if \(X_{IND} \sim X\) and \(W_{IND} \sim W\), we have

\[E[e^{-r(X-cW)}] \leq (\geq) E[e^{-rX_{IND}}]E[e^{cW_{IND}}].\]  

(53)

Then, (53) implies

\[\rho \leq (\geq) \rho_{IND}.\]  

(54)

(if they exist) or equivalently that an upper bound for the ruin probability is lower (higher) in the dependent setting than in the independent case.

In the numerical example, \((X, cW)\) are positively associated in model A, negatively associated in model B and independent in model IND. Using (58), we obtain as expected the following result regarding the Lundberg adjustment coefficients.
\[ \rho_A \leq \rho_{\text{IND}} \leq \rho_B \]
\[ -0.16667 \leq -0.13398 \leq -0.11092 \]
which implies that \( e^{\rho_A u} \leq e^{\rho_{\text{IND}} u} \leq e^{\rho_B u} \) for \( u \geq 0 \).

In fact, when \( (-X, cW) \) are negatively (positively) associated, the increments of the surplus process \( (cW_j - X_j) \) for \( j \in \mathbb{N}_+ \) are more (less) dispersed than in the independent case. Therefore, more (less) volatile is the behavior of the surplus process, more (less) likely it will cross the barrier level of 0 which is in the same line with (54).

**Acknowledgements**

The authors wish to thank Gordon Willmot and an anonymous referee for their helpful comments. Partial funding in support of this work was provided by the Natural Sciences and Engineering Research Council of Canada and by a joint grant from the Chaire en Assurance L’Industrielle-Alliance (Université Laval).

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