A note on the computation of sharp numerical bounds for the distribution of the sum, product or ratio of dependent risks

Hélène Cossette, Marie-Pier Côté, Mélina Mailhot, Etienne Marceau

Abstract

In this paper, an approximation method for computing numerically the cumulative distribution function of the sum of \( d \) random variables is developed. The method leads to numerical bounds for the distribution of the sum of dependent risks. The bounds are fast to compute and converge to the exact value if the joint probability density function exists. They also allow to evaluate sharp numerical bounds on the Value-at-Risk measure. Moreover, the fact that the approximation is deterministic, hence without uncertainty on the resulting values, is an advantage over MC simulation techniques. Applications in actuarial science and finance illustrate the accuracy of the procedure. We also present analogous bounds for the distribution of the product or the ratio of two random variables, which can be useful for actuarial or financial applications.

1. Introduction

Situations where the distribution of a sum of dependent risks needs to be computed are frequent in actuarial science, in quantitative risk management, in statistics and in applied probability in general. Calculation of overall capital charge for a portfolio of risks, evaluation and analysis of risk measures for decision making, and strategic planning are circumstances where knowledge of the cumulative distribution function (cdf) of dependent random variables (rvs) is crucial. Assuming independence between risks allows a fast computation of their aggregate distribution, but may be inappropriate and induce wrong results. Many quantities of interest, such as risk measures, are computed using the distribution of aggregated risks.

Several techniques have been developed to approximate or to calculate bounds if there is no closed-form expression for the cdf of the sum of dependent rvs. [3] proposes a numerical method for the evaluation of the cdf of a sum of rvs. Their approach is based on the discretization of the marginals (see e.g. [12]) and leads to lower and upper bounds on the exact value. [1] developed the AEP algorithm for fast computation of the distribution of the aggregated risks. Their numerical method is deterministic and provides accurate point approximations for the cdf of a sum of rvs. [5] suggests methods to approximate the distribution of dependent risks, based on compound Poisson distributions.

In this paper, we introduce a method inspired by the AEP algorithm. We develop lower and upper numerical bounds for the cdf of a sum of rvs that converge to the exact value when the joint probability density function (pdf) exists. The

numerical bounds are fast to compute and have many advantages in comparison to simulation techniques. It allows the computation of two numerical bounds for the Value-at-Risk (VaR) of the sum of dependent risks. Our approach can also be seen as a complement to the numerical approach proposed by [1].

Products and ratios of rvs are also interesting in actuarial and financial applications. A product of two rvs \( X_1 \) and \( X_2 \) could arise in evaluating the accumulated value of an investment, or the payment from a reinsurance agreement (if \( X_1 \) is the total loss and \( X_2 \) is for example an indicator of some event, or a percentage based on an index). In an insurance context, it is of interest to model the loss ratio, defined as the total loss divided by the income over the period. Both components can be dependent, especially due to reinsurance agreements and/or return on assets. This motivates the extension of our method to evaluate bounds on the cdf of product or ratio of rvs.

The outline of the paper is as follows. In Section 2, we present the algorithm in the bivariate and trivariate cases, and then we show expressions for the general case, along with the proof of convergence. In Section 2.5, numerical bounds on the risk measure VaR are developed. Section 3 presents many applications and comparison of our approach with the one of [1]. Finally, the last two sections extend the method to the cdf of the product or the ratio of two rvs, respectively.

2. Description of the method

This section aims to describe the algorithm proposed and shows its easiness of implementation, in two and three dimensions. We first briefly explain the AEP algorithm and our motivation for developing our method. Then, we present the bivariate and trivariate cases. The results are extended to higher dimensions in Section 2.4.

2.1. AEP algorithm and motivation

Let us start by describing the AEP method from [1]. The main idea of the AEP algorithm is to calculate the cdf of the sum of \( d \) rvs, evaluated at a constant \( s \). This is equivalent to evaluating the probability that the random vector \( (X_1, \ldots, X_d) \) takes values in a \( d \)-simplex, which is represented by the region in the space of \( X_1, \ldots, X_d \) such that \( X_1 + \cdots + X_d \leq s \). In the first iteration, we calculate the probability in a hypercube of the same dimension as the number of aggregated random variables, with edge length equal to \( s/\alpha_{AEP} \), and \( \alpha_{AEP} \in [1/d, 1) \). This first step creates new smaller \( d \)-simplexes around the hypercube, so new hypercubes are added and removed, until the initial simplex is almost totally filled. The reader is referred to [1] for the detailed procedure. Throughout this paper, the rvs \( X_1, \ldots, X_d \) are assumed to be non-negative as is often the case in actuarial applications. However, [1] also discussed the extension to random vector bounded from below.

For instance, in two dimensions, Fig. 1 illustrates the steps followed by the AEP algorithm, after one and two iterations (left and right figure respectively), for \( s = 1 \) and \( \alpha_{AEP} = 0.7 \). On these figures, the probabilities in the gray squares are added, while the probability in the black square is subtracted. At the end of the second iteration, nine new triangles are formed and they will be approximated by squares in the third iteration, and so on. As the number of iterations increases, the new triangles are smaller and the accuracy of the approximation is improved. This algorithm converges quickly for clever values of \( \alpha_{AEP} \); [1] provides optimal values for this parameter (e.g. \( \frac{2}{3} \) for \( d = 2 \) and \( \frac{1}{2} \) for \( d = 3 \)).

From Fig. 1, one can see that the approximation may either underestimate or overestimate the true value of the cdf when \( \alpha_{AEP} > 1/d \). In order to compute a conservative value of some risk measures in insurance or finance applications, it is of interest to understand if the approximation lies below or above the true value. For \( \alpha_{AEP} = 0.5 \) in two dimensions, the approximation will always lie below the true value, as illustrated in Fig. 2. This corresponds to our lower bound in two dimensions, when we reformulate the problem in terms of rectangles. We also propose an upper bound based on rectangles.
that cover entirely the region of interest. This approach provides numerical bounds that are more precise as the number of rectangles increases. Similarly as a Riemann sum, there is no overlapping of the rectangles, either for the lower or the upper bound. We proceed analogously in \(d\) dimensions, using \(d\)-dimensional boxes.

The method suggested in this paper, as well as in [1], is deterministic and hence there is no uncertainty on the final estimates, unlike the Monte Carlo (MC) and quasi-Monte Carlo (QMC) techniques. Also, it does not necessitate a smooth or analytic pdf. The calculation is done quickly and produces exact lower and upper bounds, getting closer to the true value as the number of rectangles increases.

2.2. Bivariate case

We consider a random pair of non-negative rvs \(X = (X_1, X_2)\) with joint cdf \(F_{X}\). We define the rv \(S\) as the sum of \(X_1\) and \(X_2\). We want to evaluate the cdf of \(S\) given by

\[
F_S(s) = \Pr[X_1 + X_2 \leq s] = \Pr[(X_1, X_2) \in \text{triangle } \Omega^2(s)],
\]

where \(\Omega^2(s)\) is the triangular region of the quadrant under the straight line from \((s, 0)\) to \((0, s)\). It is simply the region \(\{(x_1, x_2)|x_1 + x_2 \leq s\}\), which is a triangle whose vertices are \((0, 0), (0, s)\) and \((s, 0)\).

We develop lower and upper bounds for the exact value of \(F_S(s)\), by covering the area of \(\Omega^2(s)\) using rectangles. The number of rectangles is \(2^m - 1\) for the lower bound and \(2^m\) for the upper bound, where the parameter \(m\) defines the chosen precision. Increasing the parameter \(m\) will improve the accuracy of the approximation by shrinking the width of the rectangles. Fig. 3 represents the method for \(m = 2\), when \(s = 1\).

**Definition 2.1.** Let \(S = X_1 + X_2\), where \(X = (X_1, X_2)\) has cdf \(F_X\), and let \(s \geq 0\).

(a) The lower bound for \(F_S(s)\) calculated with precision parameter \(m \in \mathbb{N}^+\), denoted \(A_S^{(l,m)}(s)\), is given by

\[
A_S^{(l,m)}(s) = \sum_{i=1}^{2^m-1} c_X^{(l,m)}(s; i),
\]
where
\[ \zeta_X^{(l,m)} (s; i) = \Pr (X \in \text{rectangle } i) = \Pr \left( \frac{(i-1)}{2^m} s < X_1 \leq \frac{i}{2^m} s, \frac{2^m + 1 - i}{2^m} s \right) \]
\[ = F_X \left( \frac{i}{2^m} s, \frac{2^m + 1 - i}{2^m} s \right) - F_X \left( \frac{(i-1)}{2^m} s, \frac{2^m + 1 - i}{2^m} s \right), \]
for \( i = 1, 2, \ldots, 2^m - 1 \).

(b) The upper bound for \( F_S (s) \) calculated with precision parameter \( m \in \mathbb{N}^+ \), denoted \( A_{(u,m)}^{(s)} (s) \), is given by
\[ A_{(u,m)}^{(s)} (s) = \sum_{i=1}^{2^m} \zeta_X^{(u,m)} (s; i), \]
where
\[ \zeta_X^{(u,m)} (s; i) = \Pr (X \in \text{rectangle } i) = \Pr \left( \frac{(i-1)}{2^m} s < X_1 \leq \frac{i}{2^m} s, \frac{2^m + 1 - i}{2^m} s \right) \]
\[ = F_X \left( \frac{i}{2^m} s, \frac{2^m + 1 - i}{2^m} s \right) - F_X \left( \frac{(i-1)}{2^m} s, \frac{2^m + 1 - i}{2^m} s \right), \]
for \( i = 1, 2, \ldots, 2^m \).

Remark 2.2. By construction, \( A_{(l,m)}^{(s)} (s) \leq F_S (s) \leq A_{(u,m)}^{(s)} (s) \), for \( s \geq 0 \) and \( m \in \mathbb{N}^+ \).

2.3. Trivariate case

We consider a random vector of non-negative rvs \( X = (X_1, X_2, X_3) \) with joint cdf \( F_X \). We define the rv \( S \) as the sum of \( X_1, X_2, \) and \( X_3 \). We want to evaluate the continuous cdf of \( S \) given by
\[ F_S (s) = \Pr [X_1 + X_2 + X_3 \leq s] = \Pr [ (X_1, X_2, X_3) \in \text{tetrahedron } \Theta^3 (s)], \]
where \( \Theta^3 (s) \) is the tetrahedron (3-simplex) with vertices at the points (0, 0, 0), (s, 0, 0), (0, s, 0) and (0, 0, s). It is the region represented by \( \{ (x_1, x_2, x_3) | x_1 + x_2 + x_3 \leq s \} \). We develop sharp lower and upper bounds for \( F_S \), by adding the probabilities contained in boxes (square prisms). Fig. 4 represents the algorithm for \( m = 1 \), when \( s = 1 \).

Definition 2.3. Let \( S = X_1 + X_2 + X_3 \), where \( X = (X_1, X_2, X_3) \) has cdf \( F_X \), and let \( s \geq 0 \).
(a) The lower bound for $F_3(s)$ calculated with precision parameter $m \in \mathbb{N}^+$ is given by

$$A_{S}^{(l,m)}(s) = \sum_{i_1=1}^{3^m-2} \sum_{i_2=1}^{3^m-1-i_1} \zeta_{\Sigma}^{(l,m)}(s; i_1, i_2),$$

where

$$\zeta_{\Sigma}^{(l,m)}(s; i_1, i_2) = \Pr \left[ X \in \text{box}(i_1, i_2) \right]$$

$$= \Pr \left[ \frac{i_1 - 1}{3^m}s < X_1 \leq \frac{i_1}{3^m}s, \frac{i_2 - 1}{3^m}s < X_2 \leq \frac{i_2}{3^m}s, \frac{3^m - i_1 - i_2}{3^m}s \leq X_3 \leq \frac{3^m - i_1 - i_2}{3^m}s \right],$$

$$= F_{\Sigma} \left( \frac{i_1}{3^m}s, \frac{i_2}{3^m}s, \frac{3^m - i_1 - i_2}{3^m}s \right) - F_{\Sigma} \left( \frac{i_1 - 1}{3^m}s, \frac{i_2 - 1}{3^m}s, \frac{3^m - i_1 - i_2}{3^m}s \right)$$

$$- F_{\Sigma} \left( \frac{i_1}{3^m}s, \frac{i_2 - 1}{3^m}s, \frac{3^m - i_1 - i_2}{3^m}s \right) + F_{\Sigma} \left( \frac{i_1 - 1}{3^m}s, \frac{i_2 - 1}{3^m}s, \frac{3^m - i_1 - i_2}{3^m}s \right),$$

for $i_1 = 1, \ldots, 3^m - 2$ and $i_2 = 1, \ldots, 3^m - 1 - i_1$.

(b) The upper bound for $F_3(s)$ calculated with precision parameter $m \in \mathbb{N}^+$ is given by

$$A_{S}^{(u,m)}(s) = \sum_{i_1=1}^{3^m} \sum_{i_2=1}^{3^m+1-i_1} \zeta_{\Sigma}^{(u,m)}(s; i_1, i_2),$$

with

$$\zeta_{\Sigma}^{(u,m)}(s; i_1, i_2) = \Pr \left[ X \in \text{box}(i_1, i_2) \right]$$

$$= \Pr \left[ \frac{i_1 - 1}{3^m}s < X_1 \leq \frac{i_1}{3^m}s, \frac{i_2 - 1}{3^m}s < X_2 \leq \frac{i_2}{3^m}s, \frac{3^m + 2 - i_1 - i_2}{3^m}s \leq X_3 \leq \frac{3^m + 2 - i_1 - i_2}{3^m}s \right],$$

$$= F_{\Sigma} \left( \frac{i_1}{3^m}s, \frac{i_2}{3^m}s, \frac{3^m + 2 - i_1 - i_2}{3^m}s \right) - F_{\Sigma} \left( \frac{i_1 - 1}{3^m}s, \frac{i_2 - 1}{3^m}s, \frac{3^m + 2 - i_1 - i_2}{3^m}s \right)$$

$$- F_{\Sigma} \left( \frac{i_1}{3^m}s, \frac{i_2 - 1}{3^m}s, \frac{3^m + 2 - i_1 - i_2}{3^m}s \right) + F_{\Sigma} \left( \frac{i_1 - 1}{3^m}s, \frac{i_2 - 1}{3^m}s, \frac{3^m + 2 - i_1 - i_2}{3^m}s \right),$$

for $i_1 = 1, \ldots, 3^m$ and $i_2 = 1, \ldots, 3^m + 1 - i_1$.

**Remark 2.4.** By construction, $A_{S}^{(l,m)}(s) \leq F_3(s) \leq A_{S}^{(u,m)}(s)$, for $s \geq 0$ and $m \in \mathbb{N}^+$. 

![Fig. 5. Illustration of the overlapping simplexes created in iteration 1 of the AEP algorithm with $\alpha = 1/3$ and $s = 1$.](image-url)
Remark 2.5. The lower bound corresponds to the AEP algorithm with \(a = 1/d\) only in the bivariate case. It is not the same process in the trivariate case. In fact, when \(d = 3\) and \(a = 1/3\), for each previous cube, 3 cubes are added while 3 cubes are removed, as the 3-simplexes created around the cube are overlapping. To illustrate, let us observe the first iteration in Fig. 5. In the right panel, one sees that three pyramids of edge length \(2/3\) are created around the cube, but these pyramids overlap, which creates three smaller pyramids. If we do not subtract the probabilities in the small pyramids, they will be included twice. This is why, even when \(a = 1/3\), the AEP algorithm necessitates to add and remove probabilities contained in cubes.

2.4. General case

The formulas presented in the previous section can be extended to the general case, for any dimension. Let \(S = \sum_{k=1}^d X_k\). Then, we are interested in evaluating bounds on the cdf of \(S\), which is

\[ F_S(s) = \Pr[S \leq s] = \Pr[(X_1, \ldots, X_d) \in \text{hyper-box}(1, \ldots, d)] \]

where \(\text{hyper-box}(1, \ldots, d)\) is the hyper-box formed by the bivariate case, which is not the same.

Definition 2.6. Let \(X = (X_1, \ldots, X_d)\) be a \(d\)-dimensional vector of non-negative random variables with joint cdf \(F_X\). Let \(s \geq 0\) and \(S = \sum_{i=1}^d X_i\). Also, denote by \(j_0, \ldots, j_N\) all the \(2^{d-1}\) vectors in \(\{0, 1\}^{d-1}\), where \(N = 2^{d-1} - 1\). For example, if \(d = 3\), we have four vectors \(j_0 = (0, 0), j_1 = (1, 0), j_2 = (0, 1), j_3 = (1, 1)\). Let \(j_{k,i}\) be the number of ones in the vector \(j_k = (j_k, j_k, \ldots, j_k, d-1)\).

(a) The lower bound for \(F_S(s)\) calculated with precision parameter \(m\) is given by

\[ A_S^{(l,m)}(s) = \sum_{i_1=1}^{d^m-(d-1)} \sum_{i_2=1}^{d^m-(d-1)+i_1} \cdots \sum_{i_{d-1}=1}^{d^m-\sum_{l=1}^{d-2} i_l} \zeta_X^{(l,m)}(s; i_1, \ldots, i_{d-1}) , \]  

where

\[ \zeta_X^{(l,m)}(s; i_1, \ldots, i_{d-1}) = \Pr[(X_1, \ldots, X_d) \in \text{hyper-box}(i_1, \ldots, i_d)] \]

\[ = \Pr \left[ \frac{i_1 - 1}{d^m} s < X_1 \leq \frac{i_1}{d^m} s, \frac{i_2 - 1}{d^m} s < X_2 \leq \frac{i_2}{d^m} s, \ldots, \frac{i_{d-1} - 1}{d^m} s < X_{d-1} \leq \frac{i_{d-1}}{d^m} s, \frac{d^m - \sum_{l=1}^{d-1} i_l}{d^m} s < X_d \right] \]

\[ = \sum_{k=0}^N (-1)^k F_X \left( \frac{i_1 - j_{k,1}}{d^m} s, \ldots, \frac{i_{d-1} - j_{k,d-1}}{d^m} s, \frac{d^m - \sum_{l=1}^{d-1} i_l}{d^m} s \right) , \]

for \(i_1 = 1, \ldots, d^m - (d - 1), i_2 = 1, \ldots, d^m - (d - 1) + 1 - i_1, \ldots, i_{d-1} = 1, \ldots, d^m - 1 - \sum_{l=1}^{d-2} i_l\).

(b) The upper bound for \(F_S(s)\) calculated with precision parameter \(m\) is given by

\[ A_S^{(u,m)}(s) = \sum_{i_1=1}^{d^m} \sum_{i_2=1}^{d^m+1-i_1} \cdots \sum_{i_{d-1}=1}^{d^m-d-2-\sum_{l=1}^{d-2} i_l} \zeta_X^{(u,m)}(s; i_1, \ldots, i_{d-1}) , \]  

where
\[ \zeta_X^{(u,m)}(s; i_1, \ldots, i_{d-1}) = \Pr\left[ (X_1, \ldots, X_d) \in \text{hyper-box } (i_1, \ldots, i_{d-1}) \right] \]

\[ = \Pr \left[ \frac{i_1 - 1}{d^m}s < X_1 \leq \frac{i_1}{d^m}s, \ \frac{i_2 - 1}{d^m}s < X_2 \leq \frac{i_2}{d^m}s, \ldots, \right. \]

\[ \left. \frac{i_{d-1} - 1}{d^m}s < X_{d-1} \leq \frac{i_{d-1}}{d^m}s, \ X_d \leq \frac{d^m + d - 1 - \sum_{i=1}^{d-1} i_i}{d^m}s \right] \]

\[ = \sum_{k=0}^{N} (-1)^k F_X \left( \frac{i_1 - j_{k,1}}{d^m}s, \ldots, \frac{i_{d-1} - j_{k,d-1}}{d^m}s, \frac{d^m + d - 1 - \sum_{i=1}^{d-1} i_i}{d^m}s \right), \]

for \( i_1 = 1, \ldots, d^m, \ i_2 = 1, \ldots, d^m + 1 - i_1, \ldots, i_{d-1} = 1, \ldots, d^m + d - 2 - \sum_{i=1}^{d-2} i_i. \)

For example, for the quadrivariate case \( X = (X_1, \ldots, X_4), \) we find that \( \zeta_X^{(l,m)}(s; i_1, i_2, i_3) \) is equal to

\[ F_X \left( \frac{i_1}{4^m}s, \frac{i_2}{4^m}s, \frac{i_3}{4^m}s, \frac{4^m - i_1 - i_2 - i_3}{4^m}s \right) - F_X \left( \frac{i_1 - 1}{4^m}s, \frac{i_2}{4^m}s, \frac{i_3}{4^m}s, \frac{4^m - i_1 - i_2 - i_3}{4^m}s \right) \]

\[ - F_X \left( \frac{i_1 - 1}{4^m}s, \frac{i_2 - 1}{4^m}s, \frac{i_3}{4^m}s, \frac{4^m - i_1 - i_2 - i_3}{4^m}s \right) + F_X \left( \frac{i_1 - 1}{4^m}s, \frac{i_2 - 1}{4^m}s, \frac{i_3 - 1}{4^m}s, \frac{4^m - i_1 - i_2 - i_3}{4^m}s \right) \]

while the expression for \( \zeta_X^{(u,m)}(s; i_1, i_2, i_3) \) is simply

\[ F_X \left( \frac{i_1}{4^m}s, \frac{i_2 - 1}{4^m}s, \frac{i_3}{4^m}s, \frac{4^m + 3 - i_3 - i_2 - i_1}{4^m}s \right) - F_X \left( \frac{i_1 - 1}{4^m}s, \frac{i_2}{4^m}s, \frac{i_3}{4^m}s, \frac{4^m + 3 - i_3 - i_2 - i_1}{4^m}s \right) \]

\[ - F_X \left( \frac{i_1 - 1}{4^m}s, \frac{i_2 - 1}{4^m}s, \frac{i_3}{4^m}s, \frac{4^m - 3 - i_3 - i_2 - i_1}{4^m}s \right) + F_X \left( \frac{i_1 - 1}{4^m}s, \frac{i_2 - 1}{4^m}s, \frac{i_3 - 1}{4^m}s, \frac{4^m + 3 - i_3 - i_2 - i_1}{4^m}s \right) . \]

These formulas are easy to apply and to explain even for \( d > 3, \) which is a major advantage over most other techniques. Numerical results will be presented in Section 3.

**Remark 2.7.** By **Definition 2.6**, it is clear that

\[ A_S^{(l,m)}(s) \leq F_S(s) \leq A_S^{(u,m)}(s), \]

for all \( s \geq 0, \ m \in \mathbb{N}^+. \) Also, we have

\[ A_S^{(l,m)}(s) \leq A_S^{(l,m+k)}(s) \leq F_S(s) \leq A_S^{(u,m+k)}(s) \leq A_S^{(u,m)}(s), \]

for all \( s \geq 0 \) and for \( m, k \in \mathbb{N}^+, \) meaning that the bounds get tighter as \( m \) increases.

In fact, **Remark 2.7** applies without any restriction on the distribution of \( X, \) meaning that the bounds are valid even when the distribution is not continuous. Moreover, the following proposition states that the bounds will ultimately converge to the true value of \( F_S(s) \) if the distribution is absolutely continuous. Note that the assumptions of continuity and boundedness in the neighborhood of the simplex diagonal were also required in the proof of convergence of the AEP algorithm in [4].
**Proposition 2.8.** Let \( s \geq 0 \) and \( X \) be a \( d \)-dimensional random vector with absolutely continuous cdf \( F_X \) and pdf \( f_X \) that is bounded in the neighborhood of the simplex diagonal (i.e. the hyperplane \( \{ \sum_{i=1}^{d} x_i = s \} \)). Then, the lower and upper bounds converge to the exact distribution function when \( m \) tends to infinity, that is

\[
\lim_{m \to \infty} \left| A_s^{(l,m)}(s) - F_s(s) \right| = 0.
\]

and

\[
\lim_{m \to \infty} \left| A_s^{(u,m)}(s) - F_s(s) \right| = 0.
\]

**Proof.** Let \( Q_{i_1,\ldots,i_{d-1}}^{(l,m)}(s) \) be the hyperbox defined by

\[
\left\{ \frac{i_1 - 1}{d^m} s < X_1 \leq \frac{i_1}{d^m} s, \ldots, \frac{i_d - 1}{d^m} s < X_d - 1 \leq \frac{i_d}{d^m} s, X_d \leq \frac{d^m - \sum_{i=1}^{d-1} i_l}{d^m} s \right\}.
\]

Then, the lower bound can be rewritten as

\[
A_s^{(l,m)}(s) = \Pr \left[ X \in \bigcup_{i_{d-1}=1}^{d^m-1} \bigcup_{i_{d-1}=1}^{d^m-\sum_{i=1}^{d-1} i_l} Q_{i_1,\ldots,i_{d-1}}^{(l,m)}(s) \right].
\]

Also, define the sub-simplex

\[
S_m^l = \left\{ 0 \leq x_1 + \cdots + x_d \leq \left( 1 - \frac{d-1}{d^m} \right) s, x_1, \ldots, x_d \geq 0 \right\}.
\]

It is easily seen from (1) that this sub-simplex is entirely contained in the hyperboxes:

\[
\bigcup_{i_{d-1}=1}^{d^m-1} \bigcup_{i_{d-1}=1}^{d^m-\sum_{i=1}^{d-1} i_l} Q_{i_1,\ldots,i_{d-1}}^{(m)} \supseteq S_m^l.
\]

Hence, we have

\[
\Pr[X \in S_m^l] - A_s^{(l,m)}(s) \leq 0. \tag{4}
\]

Recall that the Lebesgue measure of a simplex of this type with edge length \( h \) is simply equal to \( \frac{h^d}{d!} \), as stated in [1]. Then, if the pdf is bounded by a constant \( c \) in the neighborhood of the simplex diagonal (i.e. the hyperplane \( \{ x_1 + \cdots + x_d = s \} \)), we have that

\[
0 \leq \Pr[X \in \Theta^d(s)] - \Pr[X \in S_m^l] = \int_{\Theta^d(s) \backslash S_m^l} f_X(x_1, \ldots, x_d) dx_1 \cdots dx_d
\]

\[
\leq c \left[ \lambda(\Theta^d(s)) - \lambda(S_m^l) \right]
\]

\[
= c \left[ \frac{s^d}{d!} - \left( 1 - \frac{d-1}{d^m} \right)^d \frac{s^d}{d!} \right]
\]

\[
= c \frac{s^d}{d!} \left[ 1 - \left( 1 - \frac{d-1}{d^m} \right)^d \right],
\]

where \( \lambda(A) \) denotes the Lebesgue measure of the set \( A \). Thus, we have

\[
\lim_{m \to \infty} \left| \Pr[X \in \Theta^d(s)] - \Pr[X \in S_m^l] \right| \leq \lim_{m \to \infty} c \frac{s^d}{d!} \left[ 1 - \left( 1 - \frac{d-1}{d^m} \right)^d \right] = 0. \tag{5}
\]

From Remark 2.7, \( F_s(s) - A_s^{(l,m)} \geq 0 \). Using both (4) and (5), we have

\[
0 \leq F_s(s) - A_s^{(l,m)}(s) = F_s(s) - \Pr[X \in S_m^l] + \Pr[X \in S_m^l] - A_s^{(l,m)}(s)
\]

\[
\leq F_s(s) - \Pr[X \in S_m^l]
\]

\[
= \Pr[X \in \Theta^d(s)] - \Pr[X \in S_m^l] \longrightarrow 0 \quad \text{if } m \to \infty.
\]
Therefore, we obtain
\[
\lim_{m \to \infty} \left| A_5^{(l,m)}(s) - F_5(s) \right| = 0.
\]
Similarly, the upper bound can be written
\[
A_5^{(u,m)}(s) = \Pr \left[ X \in \bigcup_{i_1=1}^{d^m} \cdots \bigcup_{i_{d-1}=1}^{d^m+d-2-\sum_{i=1}^{d-2} i_l} Q_{i_1,\ldots,i_{d-1}}^{(u,m)}(s) \right],
\]
where
\[
Q_{i_1,\ldots,i_{d-1}}^{(u,m)}(s) = \left\{ \frac{i_1-1}{d^m} s < X_1 \leq \frac{i_1}{d^m} s, \ldots, \frac{i_{d-1}-1}{d^m} s < X_{d-1} \leq \frac{i_{d-1}}{d^m} s, X_d \leq \frac{d^m+d-1-\sum_{i=1}^{d-1} i_l}{d^m} s \right\}.
\]
Define the simplex
\[
S_m^u = \left\{ 0 \leq x_1 + \cdots + x_d \leq \left( 1 + \frac{d-1}{d^m} \right) s, x_1, \ldots, x_d \in [0, s] \right\}.
\]
It is easily seen from (2) that the hyperboxes are entirely contained in this simplex:
\[
\bigcup_{i_1=1}^{d^m} \cdots \bigcup_{i_{d-1}=1}^{d^m+d-2-\sum_{i=1}^{d-2} i_l} Q_{i_1,\ldots,i_{d-1}}^{(u,m)}(s) \subseteq S_m^u.
\]
Clearly, we have that
\[
A_5^{(u,m)}(s) - \Pr[X \in S_m^u] \leq 0. \tag{6}
\]
Then, if the pdf is bounded by any constant \(c\) in the neighborhood of the simplex diagonal (i.e. the hyperplane \(\{x_1 + \cdots + x_d = s\}\)), we have that
\[
0 \leq \Pr[X \in S_m^u] - \Pr[X \in \mathcal{S}^d(s)] = \int_{S_m^u \setminus \mathcal{S}^d(s)} f_X(x_1, \ldots, x_d)dx_1 \ldots dx_d
\leq c[\lambda(S_m^u) - \lambda(\mathcal{S}^d(s))]
\leq c \left[ \frac{d}{d!} \right] \frac{s^d}{d!} \left[ 1 - \left( 1 + \frac{d-1}{d^m} \right)^d \right]
= c \frac{s^d}{d!} \left[ 1 - \left( 1 + \frac{d-1}{d^m} \right)^d \right].
\]
Thus, we have
\[
\lim_{m \to \infty} \left| \Pr[X \in S_m^u] - \Pr[X \in \mathcal{S}^d(s)] \right| \leq \lim_{m \to \infty} c \frac{s^d}{d!} \left[ 1 - \left( 1 + \frac{d-1}{d^m} \right)^d \right] = 0. \tag{7}
\]
By definition, \(A_5^{(l,m)}(s) - F_5(s) \geq 0\). Using both (6) and (7), we have
\[
0 \leq A_5^{(u,m)}(s) - F_5(s) = A_5^{(u,m)}(s) - \Pr[X \in S_m^u] + \Pr[X \in S_m^u] - F_5(s)
\leq \Pr[X \in S_m^u] - F_5(s)
= \Pr[X \in S_m^u] - \Pr[X \in \mathcal{S}^d(s)] \to 0 \quad \text{if } m \to \infty.
\]
Therefore, we get
\[
\lim_{m \to \infty} \left| A_5^{(u,m)}(s) - F_5(s) \right| = 0. \quad \square
\]

**Remark 2.9.** The assumption of absolute continuity of the distribution of \(X\) is strong. In fact, one only needs to have no discontinuity on the simplex diagonal in order to get convergence of the bounds. In Section 3.3, we present an example where the joint pdf does not exist but the bounds still converge.
2.5. Sharp numerical bounds on Value-at-Risk

In quantitative risk management, risk measures are usually of prime interest. Knowledge of the cdf of the aggregated risks allows the computation of the risk measure Value-at-Risk at level $\kappa$, $0 \leq \kappa < 1$, of $S$, defined by

$$\text{VaR}_\kappa (S) = \inf \{ x \in \mathbb{R}, F_S(x) \geq \kappa \}.$$ 

For $0 < \kappa < 1$, we define the following approximations of $\text{VaR}_\kappa (S)$:

$$\hat{\text{VaR}}_{\kappa}^{(u,m)} (S) = \inf \{ s \in \mathbb{R}, A_S^{(u,m)} (s) \geq \kappa \}, \quad (8)$$

and

$$\hat{\text{VaR}}_{\kappa}^{(l,m)} (S) = \inf \{ s \in \mathbb{R}, A_S^{(l,m)} (s) \geq \kappa \}. \quad (9)$$

Based on (3), we have

$$\hat{\text{VaR}}_{\kappa}^{(u,m)} (S) \leq \hat{\text{VaR}}_{\kappa}^{(l,m+k)} (S) \leq \text{VaR}_\kappa (S) \leq \hat{\text{VaR}}_{\kappa}^{(l,m+k)} (S) \leq \hat{\text{VaR}}_{\kappa}^{(l,m)} (S),$$

for all $0 < \kappa < 1$ and for $m, k \in \mathbb{N}^+$. 

**Corollary 2.10.** Let $X$ be a random vector with bounded pdf. Then, both approximations defined in (8) and (9) converge to $\text{VaR}_\kappa (S)$ when $m$ tends to infinity:

$$\lim_{m \to \infty} \left| \hat{\text{VaR}}_{\kappa}^{(l,m)} (S) - \text{VaR}_\kappa (S) \right| = 0,$$

and

$$\lim_{m \to \infty} \left| \hat{\text{VaR}}_{\kappa}^{(u,m)} (S) - \text{VaR}_\kappa (S) \right| = 0.$$

**Proof.** This lemma is a direct consequence of Proposition 2.8. □

3. Numerical illustrations

In this section, we test our algorithm on some multivariate risks of actuarial or financial interest. We will first compare our results from those obtained with the AEP algorithm, in the bivariate, trivariate and quadrivariate cases. Then, we consider four different dependence structures for a random pair and we show that our method accurately reflects the differences in the aggregate distribution. In the third example, we consider the Marshall–Olkin distribution, which does not have a pdf, and in the fourth example, we show that our method is useful when the dependence structure is very complex, as in the nested Archimedean copula case. The last example is based on the Downton–Moran multivariate exponential distribution, where we can conveniently compare our results with the exact values of the cdf of the sum $S$. Comments on the efficiency of the method are given in Section 3.6.

3.1. Comparison with AEP algorithm

Let us consider the setting of one of the illustrations presented in Section 6 of [1]. More precisely, we use the four dimensional vector $(X_1, X_2, X_3, X_4)$ with Pareto(θ, 1) marginals with tail parameters $\theta_1 = 0.9$, $\theta_2 = 1.8$, $\theta_3 = 2.6$, $\theta_4 = 3.3$, respectively, so that $f_S(x) = \theta_i/(1 + x)^{\theta_i+1}$, $x > 0$. The dependence structure is defined by the multivariate Clayton copula given by

$$C_\delta(u_1, \ldots, u_d) = (u_1^{-\delta} + \cdots + u_d^{-\delta} - d + 1)^{-1/\delta},$$

with dependence parameter $\delta > 0$ and $u_i \in [0, 1]$, $i = 1, \ldots, d$ (see e.g. [13]).

Tables 1–3 display the resulting lower and upper bounds for $F_S$ for two to four dimensions, and the approximation using the AEP algorithm. The number of iterations of the AEP algorithm was chosen in [1] to make sure the computational time is reasonable. We did the same by reducing the degree of precision of our algorithm, through the parameter $m$, when increasing the dimensions. In 2 dimensions and for $s = 100$, 10 000, 1 000 000, the lower and upper bounds both converge to the value obtained with the AEP algorithm, which implies that this value can be considered as the exact value. In 3 dimensions, the lower and upper bounds are equal to the AEP approximation for large values of $s$, but larger intervals are provided for smaller values of $s$. The results displayed in Table 3 highlight an advantage of the numerical bounds approach: the approximation using the AEP algorithm is underestimating the true value, and falls below the lower bound. The AEP algorithm does not give an idea of the precision of the approximation, nor does it indicate if the true value is underestimated or overestimated.
Table 1  
Lower and upper bounds for the sum of two Pareto marginals linked by a Clayton copula with $\delta = 1.2$.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$A^{(l,20)}_1(s)$</th>
<th>$A^{(u,20)}_1(s)$</th>
<th>AEP (14 iterations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3158349</td>
<td>0.3158351</td>
<td>0.3158350</td>
</tr>
<tr>
<td>100</td>
<td>0.9836904</td>
<td>0.9836904</td>
<td>0.9836904</td>
</tr>
<tr>
<td>10000</td>
<td>0.9997487</td>
<td>0.9997487</td>
<td>0.9997487</td>
</tr>
<tr>
<td>100000</td>
<td>0.9999960</td>
<td>0.9999960</td>
<td>0.9999960</td>
</tr>
</tbody>
</table>

Table 2  
Lower and upper bounds for the sum of three Pareto marginals linked by a Clayton copula with $\delta = 0.4$.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$A^{(l,3)}_1(s)$</th>
<th>$A^{(u,3)}_1(s)$</th>
<th>AEP (12 iterations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1908269</td>
<td>0.1908917</td>
<td>0.1908593</td>
</tr>
<tr>
<td>100</td>
<td>0.9836572</td>
<td>0.9836619</td>
<td>0.9836595</td>
</tr>
<tr>
<td>10000</td>
<td>0.9997487</td>
<td>0.9997487</td>
<td>0.9997487</td>
</tr>
<tr>
<td>100000</td>
<td>0.999996</td>
<td>0.999996</td>
<td>0.999996</td>
</tr>
</tbody>
</table>

Table 3  
Lower and upper bounds for the sum of four Pareto marginals linked by a Clayton copula with $\delta = 0.2$.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$A^{(l,4)}_1(s)$</th>
<th>$A^{(u,4)}_1(s)$</th>
<th>AEP (7 iterations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8327143</td>
<td>0.8348589</td>
<td>0.8334475</td>
</tr>
<tr>
<td>100</td>
<td>0.9835249</td>
<td>0.9837040</td>
<td>0.9834122</td>
</tr>
<tr>
<td>10000</td>
<td>0.9979798</td>
<td>0.9980012</td>
<td>0.9979503</td>
</tr>
<tr>
<td>100000</td>
<td>0.9997461</td>
<td>0.9997488</td>
<td>0.9997423</td>
</tr>
</tbody>
</table>

Table 4  
Lower and upper bounds for $F_\bar{S}(s)$, where the marginals are Lognormal(2,1) and Pareto(1.5,0.5).

<table>
<thead>
<tr>
<th>$s$</th>
<th>Countermonotonic</th>
<th>Clayton $\tau = 0.75$</th>
<th>Gumbel $\tau = 0.5$</th>
<th>Comonotonic</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>(0.5997669, 0.5997676)</td>
<td>(0.5955528, 0.5955531)</td>
<td>(0.5987150, 0.5987153)</td>
<td>(0.6022625, 0.6022629)</td>
</tr>
<tr>
<td>40</td>
<td>(0.9529500, 0.9529501)</td>
<td>(0.9447841, 0.9447842)</td>
<td>(0.9472098, 0.9472099)</td>
<td>(0.9463561, 0.9463562)</td>
</tr>
<tr>
<td>80</td>
<td>(0.9908983, 0.9908983)</td>
<td>(0.9892857, 0.9892857)</td>
<td>(0.9885533, 0.9885533)</td>
<td>(0.9881259, 0.9881259)</td>
</tr>
</tbody>
</table>

Our method allows to measure the precision of the approximation and gives a better idea of the true value for dimensions higher than 3.

3.2. Bivariate case with various dependence structures

We consider the random pair $(X_1, X_2)$ whose joint distribution is successively defined with four different copulas, namely the counter-monotonic copula, the Clayton copula with Kendall’s tau equal to 0.75, the Gumbel copula with Kendall’s tau equal to 0.5, and the comonotonic copula. The marginals are $X_1 \sim \text{LNorm}(2,1)$ and $X_2 \sim \text{Pareto}(1.5,0.5)$. It implies that $E[S] \approx 13.18$. For these different dependence structures, we obtain, with $m = 20$, the values displayed in Table 4. As can be seen, the method is accurate enough to capture the differences in the dependence structures.

3.3. Marshall–Olkin bivariate exponential distribution

The Marshall–Olkin bivariate exponential distribution is a common shock model which arises in two-dimensional Poisson processes (see [10]). This type of model is appropriate, for example in an insurance context, when the occurrence of a shock causes losses of different types. Such a common shock results in losses of each type to be dependent. It is an interesting dependence structure to use in modeling joint survival of individuals or groups. [9] also used this distribution to model the time to default for counterparties in a portfolio. If the random pair $X = (X_1, X_2)$ follows the bivariate Marshall–Olkin exponential distribution, its survival function is

$$F_S(x_1, x_2) = \exp(-(\lambda_1 + \lambda_0)x_1 - (\lambda_2 + \lambda_0)x_2 + \lambda_0 \min(x_1, x_2)).$$

While this distribution has full support, there is a singularity representing the common shock component. Therefore, this application illustrates that our method may be used with any type of distribution for which the cdf has an explicit expression. We consider the particular case where $\lambda_1 = 0.1$, $\lambda_2 = 1/3$, $\lambda_0 = 0.25$, hence the mean of $S$ is $(\lambda_1 + \lambda_0)^{-1} + (\lambda_2 + \lambda_0)^{-1} \approx 4.57$. The values of the numerical bounds for the cdf of $F_S$ are shown in Table 5 with $m = 20$. Results are very accurate: the lower and upper bounds are equal when considering up to 5 decimals, for different values of $s$. 


Table 5
Lower and upper bounds for $F_S(s)$, where $X$ is a bivariate Marshall–Olkin exponential.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$A_S^{(u, 20)}(s)$</th>
<th>$A_S^{(u, 20)}(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.6446745</td>
<td>0.6446748</td>
</tr>
<tr>
<td>10</td>
<td>0.9085079</td>
<td>0.9085081</td>
</tr>
<tr>
<td>15</td>
<td>0.9788823</td>
<td>0.9788824</td>
</tr>
<tr>
<td>20</td>
<td>0.9953940</td>
<td>0.9953941</td>
</tr>
</tbody>
</table>

Table 6
Lower and upper bounds for $F_S(s)$, for nested Clayton copula and three exponential marginals.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$A_S^{(u, 7)}(s)$</th>
<th>$A_S^{(u, 7)}(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2062971</td>
<td>0.2064639</td>
</tr>
<tr>
<td>3</td>
<td>0.5081770</td>
<td>0.5085220</td>
</tr>
<tr>
<td>10</td>
<td>0.9582851</td>
<td>0.9584525</td>
</tr>
<tr>
<td>15</td>
<td>0.9960095</td>
<td>0.9960362</td>
</tr>
</tbody>
</table>

Table 7
Lower and upper bounds for $F_S(s)$, for nested Clayton copula and four exponential marginals.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$A_S^{(u, 4)}(s)$</th>
<th>$A_S^{(u, 4)}(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.3354239</td>
<td>0.3386440</td>
</tr>
<tr>
<td>5</td>
<td>0.6609468</td>
<td>0.6659192</td>
</tr>
<tr>
<td>10</td>
<td>0.9275160</td>
<td>0.9306159</td>
</tr>
<tr>
<td>20</td>
<td>0.9991839</td>
<td>0.9992730</td>
</tr>
</tbody>
</table>

3.4. Nested Archimedean copula

Unexchangeable copula models are useful in actuarial science, quantitative risk management and statistics. Nested Archimedean copulas, as defined in [8], are flexible models that were recently used in this domain, for example for pricing collateralized debt obligations (CDOs) in [7], [11] and [6] present sampling algorithms for these models.

We consider the random vector $(X_1, X_2, X_3)$ whose joint distribution is defined with a nested Clayton copula

$$C(u_1, u_2, u_3) = \psi_1 \left[ \psi_1^{-1}(u_1) + \psi_1^{-1}(\psi_2(u_2)u_3) \right],$$

with $\psi(t) = (1 + t)^{-1/\theta}$ and $\theta_1 = 2, \theta_2 = 5$. The marginals are exponentials, with $E[X_1] = 2, E[X_2] = 1$ and $E[X_3] = 2/3$. Then, the expectation of $S$ is $2/3$.

The values of the numerical bounds for the cdf $F_S$ are shown in Table 6 with $m = 7$. We note that the complexity of the dependence structure does not affect the accuracy of our method, while performing the convolution by integrating the pdf associated to the copula looks unfeasible. Also, the bounds on the risk measure VaR at level 0.99 are $\text{VaR}_{0.99}(S) = 13.08726$ and $\text{VaR}_{0.99}^{(l, 7)}(S) = 13.09923$.

The method works as well if we add a fourth exponentially distributed rv, with mean 1/2. The joint distribution of the four rvs can be defined in terms of the following nested Clayton copula:

$$C(u_1, u_2, u_3, u_4) = \psi_1 \left[ \psi_1^{-1}(\psi_2(u_2) \psi_1^{-1}(u_1)) + \psi_1^{-1}(\psi_3(u_3) \psi_1^{-1}(u_1)) \right],$$

with $\psi(t) = (1 + t)^{-1/\theta}$ and $\theta_1 = 2, \theta_2 = 5, \theta_3 = 8$. The values of the numerical bounds for the cdf $F_S$ are shown in Table 7 with $m = 4$. Also, the bounds on the risk measure VaR are $\text{VaR}_{0.99}^{(u, 4)}(S) = 14.57368$ and $\text{VaR}_{0.99}^{(l, 4)}(S) = 14.74548$.

3.5. Downton–Moran multivariate exponential

We consider a random vector $X = (X_1, \ldots, X_d)$ with cdf defined by

$$F_X(x_1, \ldots, x_d) = \sum_{k=0}^{\infty} (1 - \gamma)^k \prod_{i=1}^{d} H \left( x_i; k + 1, \frac{\beta}{1 - \gamma} \right),$$

where $\gamma \in [0, 1)$ is Pearson’s correlation and $H(x; n, \beta)$ is the cdf of an Erlang rv evaluated at $x$, with parameters $n \in \mathbb{N}^+$ and $\beta$. For this Downton–Moran multivariate distribution, the marginals are all exponentially distributed with mean $1/\beta$. It is straightforward to show that the distribution of $S = X_1 + \cdots + X_d$ is explicitly given by

$$F_S(x) = \sum_{k=0}^{\infty} (1 - \gamma)^k H \left( x; (k + 1) d, \frac{\beta}{1 - \gamma} \right).$$
Calculation of the volume was performed using the algorithm if $\alpha$ iteration of the AEP algorithm with showing the number of cubes in which the probabilities are computed with the corresponding volume if we use up to 8 algorithm with $\alpha$ and $d$ for the $\text{VaR}$ of the sum of two to four rvs are available from the authors. Computation time could be improved greatly by using another language. $\text{R}$ uniroot() minute. This latter computation necessitates an optimization process, and we simply used $(\text{takes} 83 \text{son a Lenovo ThinkPad (2.60 GHz Intel Core i5 CPU, 8 GB RAM)},$ while the same calculation with the AEP algorithm $(n = 10)$ takes 45 s. Using our proposed method, the time necessary to evaluate $\text{VaR}^{(1)}(S)$ in Section 3.4 is less than one minute. This latter computation necessitates an optimization process, and we simply used uniroot() in R. Note that the computation time could be improved greatly by using another language. $\text{R}$ programs for calculating lower and upper bounds for the cdf or the $\text{VaR}$s of the sum of two to four rvs are available from the authors.

In Tables 8–10, we show the numerical bounds for $F_s(s)$ with $d = 2, 3, 4$ along with the exact value and the approximation obtained with the AEP algorithm, using $\gamma = 0.1$ and $\beta = 0.5$. The AEP algorithm was implemented with the optimal parameters $\alpha_{\text{AEP}}$ (which are 2/3, 1/2 and 2/5 for $d = 2, 3, 4$ respectively), as recommended by [1]. In this case again, we see that our bounds offer good precision.

3.6. Comments

The calculations for the numerical bounds in the previous sections were coded very simply using the $\text{R}$ Project for Statistical Computing. Computation time varies with the joint distribution of the rvs, the choice of $m$ and the dimensions, but it is usually fast. For Section 3.1, calculation of a lower bound with $m = 8$ using a Clayton copula and 3 Pareto marginals takes 83 s on a Lenovo ThinkPad (2.60 GHz Intel Core i5 CPU, 8 GB RAM), while the same calculation with the AEP algorithm $(n = 10)$ takes 45 s. Using our proposed method, the time necessary to evaluate $\text{VaR}^{(1,7)}(S)$ in Section 3.4 is less than one minute. This latter computation necessitates an optimization process, and we simply used uniroot() in R. Note that the computation time could be improved greatly by using another language. $\text{R}$ programs for calculating lower and upper bounds for the cdf or the $\text{VaR}$s of the sum of two to four rvs are available from the authors.

When $d = 2$, the lower bound calculated with $m$ is exactly the same as the value of the AEP algorithm with $m$ iterations and $\alpha_{\text{AEP}} = 1/2$. For both methods, one needs to compute the probability in $2^m - 1$ squares or rectangles. However, when $d = 3$, the two methods are not equivalent. To obtain approximately the same value, the number of iterations of the AEP algorithm with $\alpha_{\text{AEP}} = 1/3$ needs to be greater than the parameter $m$ for the lower bound. Table 11 illustrates this by showing the number of cubes in which the probabilities are computed with the corresponding volume if we use up to 8 iterations of the AEP algorithm with $\alpha = 1/3$. Note that the volume of the pyramid we are interested in is $s/6 = 0.16666s$. Calculation of the volume was performed using the algorithm if $X$ is uniformly distributed on the unit cube. In comparison, the number of boxes used for the computation of the lower bound with parameter $m$ is also shown in Table 11, along with the volume covered in percentage of $s$. Clearly, the AEP algorithm is not effective compared to our method for calculating a lower bound. In four dimensions, this phenomenon is even more important. The AEP algorithm is really more useful when the optimal $\alpha_{\text{AEP}}$ is used, as can be seen by the volume converging quickly to 1/6 in Table 11, but in that case one cannot tell if the result is lower or greater than the true value (some of the volume is outside the pyramid of interest).

### Table 8
Lower and upper bounds for the sum of 2 rvs with Downton–Morand distribution.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$A_{S}^{(10)}(s)$</th>
<th>$A_{S}^{(15)}(s)$</th>
<th>exact $F_{S}(s)$</th>
<th>AEP (8 iterations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.9541031</td>
<td>0.9541084</td>
<td>0.9541057</td>
<td>0.9541057</td>
</tr>
<tr>
<td>15</td>
<td>0.9930414</td>
<td>0.9930426</td>
<td>0.9930420</td>
<td>0.9930420</td>
</tr>
</tbody>
</table>

### Table 9
Lower and upper bounds for the sum of 3 rvs with Downton–Morand distribution.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$A_{S}^{(7)}(s)$</th>
<th>$A_{S}^{(15)}(s)$</th>
<th>exact $F_{S}(s)$</th>
<th>AEP (7 iterations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.8675767</td>
<td>0.8679315</td>
<td>0.8677543</td>
<td>0.8677396</td>
</tr>
<tr>
<td>15</td>
<td>0.9699873</td>
<td>0.9701098</td>
<td>0.9700486</td>
<td>0.9700437</td>
</tr>
</tbody>
</table>

### Table 10
Lower and upper bounds for the sum of 4 rvs with Downton–Morand distribution.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$A_{S}^{(4)}(s)$</th>
<th>$A_{S}^{(15)}(s)$</th>
<th>exact $F_{S}(s)$</th>
<th>AEP (5 iterations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.7382982</td>
<td>0.7455152</td>
<td>0.7419403</td>
<td>0.7392180</td>
</tr>
<tr>
<td>35</td>
<td>0.9993878</td>
<td>0.9994459</td>
<td>0.9994176</td>
<td>0.9994013</td>
</tr>
</tbody>
</table>

### Table 11
Number of cubes or boxes needed and volume covered for the AEP algorithm with $m$ iterations and the lower bound with parameter $m$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>AEP $\alpha^{*} = 1/2$</td>
<td># cubes</td>
<td>5</td>
<td>21</td>
<td>85</td>
<td>341</td>
<td>1365</td>
<td>5461</td>
</tr>
<tr>
<td>AEP $\alpha = 1/3$</td>
<td># cubes</td>
<td>7</td>
<td>43</td>
<td>259</td>
<td>1555</td>
<td>9331</td>
<td>55987</td>
</tr>
<tr>
<td>Lower Bound</td>
<td># boxes</td>
<td>28</td>
<td>325</td>
<td>3160</td>
<td>2916</td>
<td>264628</td>
<td>2388205</td>
</tr>
</tbody>
</table>

In Tables 8–10, we show the numerical bounds for $F_s(s)$ with $d = 2, 3, 4$ along with the exact value and the approximation obtained with the AEP algorithm, using $\gamma = 0.1$ and $\beta = 0.5$. The AEP algorithm was implemented with the optimal parameters $\alpha_{\text{AEP}}$ (which are 2/3, 1/2 and 2/5 for $d = 2, 3, 4$ respectively), as recommended by [1]. In this case again, we see that our bounds offer good precision.

3.6. Comments

The calculations for the numerical bounds in the previous sections were coded very simply using the $\text{R}$ Project for Statistical Computing. Computation time varies with the joint distribution of the rvs, the choice of $m$ and the dimensions, but it is usually fast. For Section 3.1, calculation of a lower bound with $m = 8$ using a Clayton copula and 3 Pareto marginals takes 83 s on a Lenovo ThinkPad (2.60 GHz Intel Core i5 CPU, 8 GB RAM), while the same calculation with the AEP algorithm $(n = 10)$ takes 45 s. Using our proposed method, the time necessary to evaluate $\text{VaR}^{(1,7)}(S)$ in Section 3.4 is less than one minute. This latter computation necessitates an optimization process, and we simply used uniroot() in R. Note that the computation time could be improved greatly by using another language. $\text{R}$ programs for calculating lower and upper bounds for the cdf or the $\text{VaR}$s of the sum of two to four rvs are available from the authors.

When $d = 2$, the lower bound calculated with $m$ is exactly the same as the value of the AEP algorithm with $m$ iterations and $\alpha_{\text{AEP}} = 1/2$. For both methods, one needs to compute the probability in $2^m - 1$ squares or rectangles. However, when $d = 3$, the two methods are not equivalent. To obtain approximately the same value, the number of iterations of the AEP algorithm with $\alpha_{\text{AEP}} = 1/3$ needs to be greater than the parameter $m$ for the lower bound. Table 11 illustrates this by showing the number of cubes in which the probabilities are computed with the corresponding volume if we use up to 8 iterations of the AEP algorithm with $\alpha = 1/3$. Note that the volume of the pyramid we are interested in is $s/6 = 0.16666s$. Calculation of the volume was performed using the algorithm if $X$ is uniformly distributed on the unit cube. In comparison, the number of boxes used for the computation of the lower bound with parameter $m$ is also shown in Table 11, along with the volume covered in percentage of $s$. Clearly, the AEP algorithm is not effective compared to our method for calculating a lower bound. In four dimensions, this phenomenon is even more important. The AEP algorithm is really more useful when the optimal $\alpha_{\text{AEP}}$ is used, as can be seen by the volume converging quickly to 1/6 in Table 11, but in that case one cannot tell if the result is lower or greater than the true value (some of the volume is outside the pyramid of interest).
4. Product of two random variables

In this section, we briefly describe how to obtain lower and upper bounds for the cdf of the product of rvs, adapting the procedure used for the sum of rvs. We are interested in the product because this simple function arises when one wants to compute e.g. accumulated values, or a payment from reinsurance agreement, and the computation of exact distribution is impossible in most cases when the rvs are dependent.

We consider the random vector \( X = (X_1, X_2) \) with joint cdf \( F_X \). Let \( T = X_1 X_2 \). Then, the cdf of \( T \) is

\[
F_T(t) = \text{Pr}[X_1 \times X_2 \leq t] = \text{Pr}[(X_1, X_2) \in \mathcal{A}(t)],
\]

where \( \mathcal{A}(t) = \{(x_1, x_2) | x_1 x_2 \leq t \} \). We develop lower and upper bounds for \( F_T \), i.e. we wish to approximate the probability that \( X \) falls in the region \( \mathcal{A}(t) \). An illustration of \( \mathcal{A}(9) \) is given by the region under the solid curve in Fig. 6.

**Definition 4.1.** Let \( T = X_1 X_2 \), where \( X = (X_1, X_2) \) has cdf \( F_X \), and let \( t \geq 0 \).

(a) The lower bound for \( F_T(t) \) calculated with precision parameter \( m \in \mathbb{N}^+ \), denoted \( A_T^{(l,m)}(t) \), is given by

\[
A_T^{(l,m)}(t) = \sum_{i=1}^{m^2} \text{Pr}[(X_1, X_2) \in \text{rectangle } i] = \sum_{i=1}^{m^2} \text{Pr} \left[ \frac{i - 1}{m} t^{1/2} < X_1 \leq \frac{i}{m} t^{1/2}, X_2 \leq \frac{m}{i} t^{1/2} \right] = \sum_{i=1}^{m^2} \left\{ F_X \left( \frac{i}{m} t^{1/2}, \frac{m}{i} t^{1/2} \right) - F_X \left( \frac{i - 1}{m} t^{1/2}, \frac{m}{i} t^{1/2} \right) \right\}.
\]

(b) The upper bound for \( F_T(t) \) calculated with precision parameter \( m \in \mathbb{N}^+ \), denoted \( A_T^{(u,m)}(t) \), is given by

\[
A_T^{(u,m)}(t) = \sum_{i=0}^{m^2-1} \text{Pr}[(X_1, X_2) \in \text{rectangle } i] = \sum_{i=0}^{m^2-1} \text{Pr} \left[ \frac{i}{m} t^{1/2} < X_1 \leq \frac{i + 1}{m} t^{1/2}, X_2 \leq \frac{m}{i} t^{1/2} \right] + \sum_{i=0}^{m^2-2} \text{Pr} \left[ \frac{m^2 - 1}{m} t^{1/2} < X_1, X_2 \leq \frac{1}{m} t^{1/2} \right] = F_{X_1} \left( \frac{t^{1/2}}{m} \right) + \sum_{i=1}^{m^2-2} \left\{ F_X \left( \frac{i + 1}{m} t^{1/2}, \frac{m}{i} t^{1/2} \right) - F_X \left( \frac{i}{m} t^{1/2}, \frac{m}{i} t^{1/2} \right) \right\} + F_{X_2} \left( \frac{t^{1/2}}{m} \right) - F_X \left( \frac{m^2 - 1}{m} t^{1/2}, \frac{t^{1/2}}{m} \right).
\]

**Fig. 6.** Illustrations of the lower and upper bounds for the distribution of the product of 2 rvs with \( m = 3 \) and \( t = 9 \).
Remark 4.2. By Definition 4.1, it is clear that

\[ A_T^{(l,m)}(t) \leq F_T(t) \leq A_T^{(u,m)}(t), \]

for all \( t \geq 0, m \in \mathbb{N}^+. \) Also, we have

\[ A_T^{(l,m)}(t) \leq A_T^{(l,m+k)}(t) \leq F_T(t) \leq A_T^{(u,m+k)}(t) \leq A_T^{(u,m)}(t), \]

for all \( t \geq 0 \) and for \( m, k \in \mathbb{N}^+ \), meaning that the bounds get tighter as \( m \) increases.

The following proposition states that the bounds converge to the exact value when the distribution of \( X \) is absolutely continuous.

**Proposition 4.3.** Let \( X = (X_1, X_2) \) be a random vector with distribution \( F_X \) absolutely continuous, and bounded pdf \( f_X \). Also, suppose that for all \( \varepsilon > 0 \), there exist constants \( M_{1,\varepsilon}, M_{2,\varepsilon} < \infty \) such that \( \Pr[X_1 > M_{1,\varepsilon}] \leq \varepsilon \) and \( \Pr[X_2 > M_{2,\varepsilon}] \leq \varepsilon \). Then, the lower and upper bounds converge to the exact distribution function when \( m \) tends to infinity, that is

\[ \lim_{m \to \infty} \left| A_T^{(l,m)}(t) - F_T(t) \right| = 0, \]

and

\[ \lim_{m \to \infty} \left| A_T^{(u,m)}(t) - F_T(t) \right| = 0. \]

**Proof.** The exact cdf is \( F_T(t) = \Pr[X \in \mathcal{A}(t)] \), where

\[ \mathcal{A}(t) = \{(x_1, x_2)|x_1 x_2 \leq t\}. \]

Suppose that for all \( \varepsilon > 0 \), there exist constants \( M_{1,\varepsilon}, M_{2,\varepsilon} < \infty \) such that \( \Pr[X_1 > M_{1,\varepsilon}] \leq \varepsilon \) and \( \Pr[X_2 > M_{2,\varepsilon}] \leq \varepsilon \). Then, the region

\[ \mathcal{A}(t, \varepsilon) = \left\{(x_1, x_2)|0 \leq x_1 \leq M_{1,\varepsilon}, 0 \leq x_2 \leq \min \left( \frac{t}{x_1}, M_{2,\varepsilon} \right) \right\} \]

is included in the region \( \mathcal{A}(t) \), and we have that

\[ F_T(t) - \Pr[X \in \mathcal{A}(t, \varepsilon)] \leq 2\varepsilon. \]

Let us start by the lower bound. Suppose we fix \( \varepsilon \) and that \( m \) is chosen large enough so that \( \frac{t^2}{m} T^{1/2} > M_{1,\varepsilon} \) and \( mt^{1/2} > M_{2,\varepsilon} \) for simplicity. Define the sub-region

\[ \mathcal{A}^{\ell}(t; \varepsilon) = \left\{(x_1, x_2)|0 \leq x_1 \leq M_{1,\varepsilon}, 0 \leq x_2 \leq \min \left( \frac{t}{x_1}, \frac{t^{1/2} m}{M_{2,\varepsilon}}, \frac{t^{1/2} m}{M_{2,\varepsilon} + \varepsilon} \right) \right\}. \]

This region is entirely contained in the rectangles used to calculate the lower bound, because the largest distance between the curve \( x_1 x_2 = t \) and the rectangles is at the point \( \left( \frac{t^{1/2} m}{M_{2,\varepsilon}}, \frac{t^{1/2} m}{M_{2,\varepsilon} + \varepsilon} \right) \), which can be equivalently expressed as

\( \left( \frac{t^{1/2} m}{M_{2,\varepsilon}}, 1 \right) \). Note that the length of each rectangle is \( \frac{t^{1/2} m}{m} \), and the slope of the curve is greater (in absolute value) close to \( x_1 = 0 \). By assumption, we know that the rectangular region \((0, t/M_{2,\varepsilon}) \times (0, M_{2,\varepsilon}) \) is completely covered when \( m \) is large, so the first uncovered region is

\[ \left\{(x_1, x_2)|\frac{t}{M_{2,\varepsilon}} < x_1 \leq \frac{t}{M_{2,\varepsilon}} + \frac{t^{1/2} m M_{2,\varepsilon}}{m}, \frac{t^{1/2} m M_{2,\varepsilon}}{m} < x_2 \leq \frac{t}{x_1} \right\}. \]

Hence, we have that

\[ A_T^{(l,m)}(t) \geq \Pr[X \in \mathcal{A}^{\ell}(t; \varepsilon)]. \]

Also note that the area of the region \( \mathcal{A}(t; \varepsilon) \) is

\[ \lambda(\mathcal{A}(t; \varepsilon)) = \int_0^{t/M_{2,\varepsilon}} M_{2,\varepsilon} dx + \int_{t/M_{2,\varepsilon}}^{M_{1,\varepsilon}} \frac{t}{x} dx = t + t \ln \left( \frac{M_{1,\varepsilon} M_{2,\varepsilon}}{t} \right), \]

for all \( t \geq 0 \) and for \( m \in \mathbb{N}^+ \), meaning that the bounds get tighter as \( m \) increases.
while the area of the sub-region is
\[
\lambda(\mathcal{A}_{m}^{l}(t; \varepsilon)) = \int_{0}^{t} \frac{t^{1/2}}{M_{1,e}} M_{2,e} dx + \int_{0}^{M_{1,e}} \frac{t^{1/2}}{M_{1,e} t^{1/2} m + M_{2,e}} dx + \int_{M_{1,e}}^{M_{2,e}} \frac{t^{1/2}}{x t^{1/2} m + M_{2,e}} dx = t \left( \frac{t^{1/2}}{t^{1/2} m + M_{2,e}} \left( 1 + \ln \left( \frac{M_{1,e} M_{2,e}}{t} \right) \right) + \ln \left( \frac{t^{1/2} m + M_{2,e}}{t^{1/2} m} \right) \right).
\]

Now, using (10), we have
\[
0 \leq F_{T}(t) - A_{T}^{(l,m)}(t) = F_{T}(t) - \Pr[X \in \mathcal{A}_{m}^{l}(t; \varepsilon)] + \Pr[X \in \mathcal{A}_{m}^{l}(t; \varepsilon)] - A_{T}^{(l,m)}(t) \leq F_{T}(t) - \Pr[X \in \mathcal{A}_{m}^{l}(t; \varepsilon)].
\]
Then, we expand again:
\[
0 \leq F_{T}(t) - A_{T}^{(l,m)}(t) \leq F_{T}(t) - \Pr[X \in \mathcal{A}(t; \varepsilon)] + \Pr[X \in \mathcal{A}(t; \varepsilon)] - \Pr[X \in \mathcal{A}_{m}^{l}(t; \varepsilon)] \\
\leq 2 \varepsilon + \int_{\mathcal{A}(t; \varepsilon) \setminus \mathcal{A}_{m}^{l}(t; \varepsilon)} f_{2}(x_{1}, x_{2}) dx_{1} dx_{2} \leq 2 \varepsilon + c \int_{\mathcal{A}(t; \varepsilon) \setminus \mathcal{A}_{m}^{l}(t; \varepsilon)} dx_{1} dx_{2}.
\]
because we assume that the pdf is bounded from above by $c$. Finally,
\[
0 \leq F_{T}(t) - A_{T}^{(l,m)}(t) \leq 2 \varepsilon + c \left( t + \ln \left( \frac{M_{1,e} M_{2,e}}{t} \right) - t \left( \frac{t^{1/2}}{t^{1/2} m + M_{2,e}} \left( 1 + \ln \left( \frac{M_{1,e} M_{2,e}}{t} \right) + \ln \left( \frac{t^{1/2} m + M_{2,e}}{t^{1/2} m} \right) \right) \right) \right) = 2 \varepsilon + c \left( \frac{t^{1/2}}{t^{1/2} m + M_{2,e}} + M_{2,e} \right) \leq t + \frac{M_{2,e}}{m} \ln \left( \frac{M_{1,e} M_{2,e}}{t} \right) + \ln \left( \frac{t^{1/2} m + M_{2,e}}{t^{1/2} m} \right) \rightarrow 0 \quad \text{as} \quad m \to \infty \text{and} \quad \varepsilon \to 0.
\]
Therefore, we obtain $\lim_{m \to \infty} A_{T}^{(l,m)}(t) - F_{T}(t) = 0$.
We proceed similarly for the upper bound. Define the region
\[
\mathcal{A}_{m}^{u}(t; \varepsilon) = \left\{(x_{1}, x_{2}) \mid 0 \leq x_{1} \leq M_{1,e}, 0 \leq x_{2} \leq \min \left( \frac{t}{x_{1}}, \frac{t^{1/2} m + M_{2,e}}{t^{1/2} m} \right) \right\}.
\]
This region entirely contains the rectangles used to calculate the upper bound, because the largest distance between the curve $x_{1} x_{2} = t$ and the rectangles is at the point $\left( \frac{t}{M_{2,e}}, \frac{t^{1/2}}{m} \right)$. The first region outside $\mathcal{A}(t; \varepsilon)$ is
\[
\left\{(x_{1}, x_{2}) \mid \frac{t}{M_{2,e}} < x_{1} \leq \frac{t}{M_{2,e}} + \frac{t^{1/2}}{m}, \frac{t}{x_{1}} < x_{2} \leq M_{2,e} \right\}.
\]
Hence, we have that
\[
A_{T}^{(u,m)}(t) \leq \Pr[X \in \mathcal{A}_{m}^{u}(t; \varepsilon)].
\]
(11)
Also note that the area of the region $\mathcal{A}_{m}^{u}(t; \varepsilon)$ is
\[
\lambda(\mathcal{A}_{m}^{u}(t; \varepsilon)) = \int_{0}^{M_{1,e}} \frac{t^{1/2}}{M_{2,e}} M_{2,e} dx + \int_{M_{1,e}}^{M_{2,e}} \frac{t}{x t^{1/2} m + M_{2,e}} dx = t + \frac{M_{2,e}}{m} + t \left( \frac{M_{2,e}}{t^{1/2} m} \right) \left( \ln \left( \frac{M_{1,e} M_{2,e}}{t} \right) + \ln \left( \frac{t^{1/2} m + M_{2,e}}{t^{1/2} m} \right) \right).
\]
Now, using (11) we have
\[
0 \leq A_{T}^{(m,u)}(t) - F_{T}(t) = A_{T}^{(l,m)}(t) - \Pr[X \in \mathcal{A}_{m}^{l}(t; \varepsilon)] + \Pr[X \in \mathcal{A}_{m}^{l}(t; \varepsilon)] - A_{T}^{(l,m)}(t) \leq \Pr[X \in \mathcal{A}_{m}^{u}(t; \varepsilon)] - F_{T}(t).
\]
Then, expanding again and using the fact that $\Pr[X \in \mathcal{A}(t; \varepsilon)] - F_T(t) \leq 0$ (from Remark 4.2),

$$0 \leq A_T^{(u,m)}(t) - F_T(t) \leq \Pr[X \in \mathcal{A}(t; \varepsilon)] - \Pr[X \in \mathcal{A}(t; \varepsilon)] + \Pr[X \in \mathcal{A}(t; \varepsilon)] - F_T(t)$$

$$= \iint_{\mathcal{A}(t; \varepsilon) \setminus \mathcal{A}(t; \varepsilon)} f_X(x_1, x_2) \, dx_1 \, dx_2.$$

Recall that we suppose the joint pdf is bounded by $c$, so we can write

$$0 \leq A_T^{(u,m)}(t) - F_T(t) \leq c \iint_{\mathcal{A}(t; \varepsilon) \setminus \mathcal{A}(t; \varepsilon)} dx_1 \, dx_2$$

$$= c \left( t + \frac{M_2 \ln t^{1/2}}{m} + t \left( 1 + \frac{M_{2,t}}{t} \ln \left( \frac{M_{1,e} M_{2,e}}{t} \right) \right) + \frac{t^{1/2} m}{t^{1/2} m + M_{2,e}} \right) - \left( 1 + \ln \left( \frac{M_{1,e} M_{2,e}}{t} \right) \right)$$

$$\longrightarrow 0 \text{ as } m \rightarrow \infty \text{ and } \varepsilon \rightarrow 0.$$

Therefore, we obtain that $\lim_{m \rightarrow \infty} |A_T^{(u,m)}(t) - F_T(t)| = 0$. □

**Example 4.4.** We consider the random pair $(X_1, X_2)$ whose joint distribution is defined by the $t$ copula with 1 degree of freedom and correlation 0.5, $X_i \sim \text{LNorm}(0.05, \sigma = 0.025)$ for $i = 1, 2$. Thus, $E[X_i] = 1.0516$. The Student $t$ copula captures tail dependence, which makes it a useful tool for modeling dependence of returns on investment. The accumulation factor for an investment over two periods can be represented as the product of the two rvs $T = X_1 X_2$. Bounds for the distribution of $T$ are shown in Table 12, with $m = 1000$.

**5. Ratio of two random variables**

Ratios of rvs are encountered in insurance applications where the interest is to model the loss ratio. However, the cdf of the ratio of dependent rvs may be very hard to compute analytically. In this section, we extend our method to handle these cases.

We consider a random vector $X = (X_1, X_2)$, for which the ratio of $X_2$ and $X_1$ is the interest. Let $W = \frac{X_2}{X_1}$, with cdf

$$F_W(w) = \Pr\left[ \frac{X_2}{X_1} \leq w \right] = \Pr[(X_1, X_2) \in \text{triangle } \Upsilon(t)],$$

where $\Upsilon(t) = \{(x_1, x_2)|x_2 \leq tx_1\}$. We want to evaluate numerical bounds on $F_W$, i.e. the probability that $X$ falls in the triangular region $\Upsilon(t)$. For example, the region $\Upsilon(2)$ is represented by the triangle under the solid line in Fig. 7 for $t = 2$.

**Definition 5.1.** Let $W = \frac{X_2}{X_1}$, where $X = (X_1, X_2)$ has cdf $F_X$ and $t \geq 0$.

(a) The lower bound for $F_W(t)$ calculated with precision parameter $m$, denoted $A_W^{(l,m)}(t)$, is given by

$$A_W^{(l,m)}(t) = \sum_{i=1}^{m^2} \Pr[(X_1, X_2) \in \text{rectangle } i]$$

$$= \sum_{i=1}^{m^2} \Pr\left[ \frac{i}{m} t < X_1 \leq \frac{i+1}{m} t, X_2 \leq \frac{i}{m^2} t^2 \right] + \Pr[X_1 > mt, X_2 \leq mt^2]$$

$$= \sum_{i=1}^{m^2} \left( F_X\left( \frac{i+1}{m} t, \frac{i}{m^2} t^2 \right) - F_X\left( \frac{i}{m} t, \frac{i}{m^2} t^2 \right) \right) + F_{X_2}(mt^2) - F_{X_2}(mt, mt^2).$$

**Table 12**

Lower and upper bounds for $F_T(t)$, for a $t$ copula with 1 degree of freedom and Lognormal marginals.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$A_T^{(l,1000)}(t)$</th>
<th>$A_T^{(u,1000)}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.05</td>
<td>0.1030833</td>
<td>0.1066007</td>
</tr>
<tr>
<td>1.10</td>
<td>0.4368020</td>
<td>0.4487682</td>
</tr>
<tr>
<td>1.15</td>
<td>0.8451108</td>
<td>0.8499882</td>
</tr>
<tr>
<td>1.20</td>
<td>0.9662964</td>
<td>0.9676655</td>
</tr>
<tr>
<td>1.25</td>
<td>0.9952943</td>
<td>0.9955490</td>
</tr>
</tbody>
</table>
Proposition 5.3. Let $X$ be a random vector with absolutely continuous distribution $F_X$ and bounded pdf $f_X$. Also, suppose that for all $\varepsilon > 0$, there exists a constant $M_\varepsilon < \infty$ such that $\Pr[X_1 > M_\varepsilon] \leq \varepsilon$. Then, the lower and upper bounds converge to the exact distribution function when $m$ tends to infinity, that is

$$
\lim_{m \to \infty} \left| A_W^{(l,m)}(t) - F_W(t) \right| = 0,
$$

and

$$
\lim_{m \to \infty} \left| A_W^{(u,m)}(t) - F_W(t) \right| = 0.
$$

Proof. The exact cdf is simply $F_W(t) = \Pr[X \in \Xi(t)]$, where

$$
\Xi(t) = \{(x_1, x_2) | 0 \leq x_1, 0 \leq x_2 \leq tx_1\}.
$$

Suppose that for all $\varepsilon > 0$, there exists a constant $M_\varepsilon < \infty$ such that $\Pr[X_1 > M_\varepsilon] \leq \varepsilon$. Then, the region

$$
W(t, \varepsilon) = \{(x_1, x_2) | 0 \leq x_1 \leq M_\varepsilon, 0 \leq x_2 \leq tx_1\},
$$

which is a triangle with vertices $(0, 0)$, $(M_1, 0)$ and $(M_1, M_1 t)$, is included in the region $\Xi(t)$, and we have that

$$
F_W(t) - \Pr[X \in W(t, \varepsilon)] \leq \varepsilon.
$$
Let us start with the lower bound. Suppose we fix $\varepsilon$ and $m$ is chosen large enough to be greater than $M_e$. Define the sub-triangle
\[
W_m(t; \varepsilon) = \left\{ (x_1, x_2) \left| \frac{t}{m} \leq x_1 \leq M_e, 0 \leq x_2 \leq tx_1 - \frac{t^2}{m} \right. \right\}.
\]
This triangle is entirely contained in the rectangles used to calculate the lower bound, meaning that $A_W^{(l,m)}(t) \geq \Pr[X \in W_m(t; \varepsilon)]$, so
\[
0 \leq F_W(t) - A_W^{(l,m)}(t) = F_W(t) - \Pr[X \in W_m(t; \varepsilon)] + \Pr[X \notin W_m(t; \varepsilon)] - A_W^{(l,m)}(t)
\leq F_W(t) - \Pr[X \notin W_m(t; \varepsilon)].
\]
Expanding again and using (12), we find
\[
0 \leq F_W(t) - A_W^{(l,m)}(t) \leq F_W(t) - \Pr[X \in W(t, \varepsilon)] + \Pr[X \notin W(t, \varepsilon)] - \Pr[X \notin W_m(t; \varepsilon)]
\leq \varepsilon + \int_{W(t, \varepsilon) \setminus W_m(t; \varepsilon)} f_X(x_1, x_2) \, dx_1 \, dx_2.
\]
Then, the assumption that the pdf is bounded by $c$ allows to write:
\[
0 \leq F_W(t) - A_W^{(l,m)}(t) \leq \varepsilon + c \int_{W(t, \varepsilon) \setminus W_m(t; \varepsilon)} dx_1 \, dx_2
= \varepsilon + c \left( \frac{M_e^2 t}{2} - \frac{(M_e - t/m)(M_e t - t^2/m)}{2} \right)
= \varepsilon + c \left( \frac{M_e t^2}{m} + \frac{t^3}{2m^2} \right) \to 0 \text{ as } m \to \infty \text{ and } \varepsilon \to 0.
\]
Therefore, we obtain $\lim_{m \to \infty} \left| A_W^{(l,m)}(t) - F_W(t) \right| = 0$.

For the upper bound, define the triangle
\[
W_u(t; \varepsilon) = \left\{ (x_1, x_2) \left| \frac{t}{m} \leq x_1 \leq M_e, 0 \leq x_2 \leq tx_1 + \frac{t^2}{m} \right. \right\}.
\]
This triangle contains entirely the rectangles used to calculate the upper bound, meaning that $A_W^{(u,m)}(t) \leq \Pr[X \in W_u(t; \varepsilon)]$, so we have
\[
0 \leq A_W^{(u,m)}(t) - F_W(t) = A_W^{(u,m)}(t) - \Pr[X \in W_u(t; \varepsilon)] + \Pr[X \notin W_u(t; \varepsilon)] - F_W(t)
\leq \Pr[X \notin W_u(t; \varepsilon)] - F_W(t).
\]
Expanding again and using (12), we find
\[
0 \leq A_W^{(u,m)}(t) - F_W(t) \leq \Pr[X \notin W_u(t; \varepsilon)] - \Pr[X \notin W(t, \varepsilon)] + \Pr[X \in W(t, \varepsilon)] - F_W(t)
\leq \int_{W_u(t; \varepsilon) \setminus W(t, \varepsilon)} f_X(x_1, x_2) \, dx_1 \, dx_2.
\]
Then, the assumption that the pdf is bounded by $c$ allows to write:
\[
0 \leq A_W^{(u,m)}(t) - F_W(t) \leq c \int_{W_u(t; \varepsilon) \setminus W(t, \varepsilon)} dx_1 \, dx_2
= c \left( \frac{(M_e + t/m)(M_e t + t^2/m)}{2} - \frac{M_e^2 t}{2} \right)
= c \left( \frac{M_e t^2}{m} + \frac{t^3}{2m^2} \right) \to 0 \text{ as } m \to \infty \text{ and } \varepsilon \to 0.
\]
Therefore, we obtain $\lim_{m \to \infty} \left| A_W^{(u,m)}(t) - F_W(t) \right| = 0$. □

Example 5.4. Let us illustrate the method by considering two rvs, $X_1 \sim \text{Gamma}(\alpha, 1)$ and $X_2 \sim \text{Exp}(1)$. Then, it is easy to show that, if the rvs are independent, the ratio $W = X_2/X_1$ follows a Pareto($\alpha, 1$). In Table 13, we compare the numerical bounds with the exact values. Table 14 shows the bounds if $X_1$ and $X_2$ are linked by a Clayton copula with dependence parameter 1/2. Note that the calculations are instantaneous.
Table 13
Exact values, and lower and upper bounds for $F_W(t)$, where $W$ is a ratio of Gamma(2,1) and Exponential(1) independent rvs.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$A_W^{(l,1000)}(t)$</th>
<th>$A_W^{(u,1000)}(t)$</th>
<th>$F_W(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.7036667</td>
<td>0.7037407</td>
<td>0.7037037</td>
</tr>
<tr>
<td>1</td>
<td>0.8749375</td>
<td>0.8750625</td>
<td>0.8750000</td>
</tr>
<tr>
<td>5</td>
<td>0.9953120</td>
<td>0.9954278</td>
<td>0.9953704</td>
</tr>
<tr>
<td>10</td>
<td>0.9992098</td>
<td>0.9992850</td>
<td>0.9992487</td>
</tr>
</tbody>
</table>

Table 14
Lower and upper bounds for $F_W(t)$, where $W$ is a ratio of Gamma(2,1) and Exponential(1) rvs linked by a Clayton copula.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$A_W^{(l,1000)}(t)$</th>
<th>$A_W^{(u,1000)}(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.7444811</td>
<td>0.7445573</td>
</tr>
<tr>
<td>1</td>
<td>0.9158122</td>
<td>0.9159085</td>
</tr>
<tr>
<td>5</td>
<td>0.9990771</td>
<td>0.9991016</td>
</tr>
<tr>
<td>10</td>
<td>0.9999298</td>
<td>0.9999370</td>
</tr>
</tbody>
</table>

6. Conclusion

The deterministic method presented in this paper has many advantages over simulation techniques. In addition to being easy to implement and to explain, it allows fast computation of sharp numerical bounds on the cdf and on the VaRs of a sum of rvs. There is no uncertainty on the results and the accuracy is defined by the user. Distributions of different transformations of rvs, such as sum, product and ratio, can be obtained using our approach. We keep for further research other extensions of this work to general increasing functions of dependent, non-negative rvs, in the same vein as [2].

Acknowledgments

This work was partially supported by the Natural Sciences and Engineering Research Council of Canada (research grants #CG053993 and #CG053934, master’s grant CGSM - 426093 - 2012), the Fonds de recherche du Québec - Nature et technologies (master’s grant 166241), and by the Chaire en actuariat de l’Université Laval (research grant FO502323). The authors are grateful to the anonymous referee for his valuable comments.

References