PENSION PLAN VALUATION AND MORTALITY PROJECTION: A CASE STUDY WITH MORTALITY DATA

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ABSTRACT

It is now well documented that human mortality globally declined during the course of the twentieth century. These mortality improvements pose a challenge for pricing and reserving in life insurance and for the management of public pension regimes. Assuming a further continuation of the stable pace of mortality decline, a Poisson log-bilinear projection model is applied to population mortality data to forecast future death rates. Then a relational model embedded in a Poisson regression approach is used to merge a dynamic mortality table based on data of a large population (in this case the Canadian province of Quebec) to mortality data of a given pension plan (here the Régie des Rentes du Québec) to create another dynamic mortality table, which can be used to make any assessments on the total costs of the pension plan. We provide at the end numerical examples that illustrate the impact of mortality improvements on a pension plan.

1. INTRODUCTION AND MOTIVATION

Mortality at adult and old ages, as demonstrated in Benjamin and Soliman (1993) and McDonald et al. (1998), reveals decreasing annual death probabilities. Life expectancy is greater than ever before and is increasing rapidly. The past 100 years have seen many improvements in life expectancy, but the pattern of improvement is changing markedly. In the first half of the twentieth century, infectious diseases were almost eradicated, and this gave massive improvements in mortality among the young ages. However, cancer and heart disease kept mortality rates stable for older people. Since then, substantial increases in longevity have been achieved at later ages. All of this poses a challenge for the planning of public retirement systems, the long-term risk management of supplemental pension plans, as well as pricing and reserving for life insurance companies. The evaluation of present values of future costs requires an appropriate mortality projection to avoid important misestimation.

Lee and Carter (1992) proposed a simple model for describing the secular change in mortality as a function of a single time index. This model is fit to historical data. The resulting estimate of the time-varying parameter then is modeled and forecast as a stochastic time series using standard Box-Jenkins methods. From this forecast of the general level of mortality, the actual age-specific rates are derived

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using the estimated age effects. The main statistical tool of Lee and Carter (1992) is least-squares estimation via singular value decomposition of the matrix of the log age-specific observed forces of mortality. This implicitly means that the errors are assumed to be homoskedastic. We refer the reader to Lee (2000) for a review of the theory and applications of this approach. Brouhns, Denuit, and Vermunt (2002a,b) and Renshaw and Haberman (2003a,b) each have implemented similar alternative approaches to mortality forecasting based on heteroskedastic Poisson error structures. A detailed account of the literature devoted to mortality projections can be found in Tuljapurkar and Boe (1998), Pitacco (2004), Wong-Fupuy and Haberman (2004), and Delwarde and Denuit (2005).

The present paper is a case study. The methodology examined in this paper is an extrapolation of past trends. All purely extrapolative forecasts assume that the future will be in some sense like the past. Some authors (see, e.g., Gutterman and Vanderhoof 1999) severely criticized this approach because it seems to ignore underlying mechanisms. As pointed out by Wilmoth (2000), such a critique is valid only insofar as such mechanisms are understood with sufficient precision to offer a legitimate alternative method of prediction. The understanding of the complex interactions of social and biological factors that determine mortality levels being still imprecise, the extrapolative approach to prediction is particularly compelling in the case of human mortality. Also, our methodology is flexible enough and can be adapted to take into account the opinions of experts in the setting of the assumptions of the forecasting model.

In a later section we use this methodology to obtain a dynamic life table allowing a better assessment of the total costs of a given private or public pension plan, in our case the Régie des Rentes du Québec (RRQ) pension plan. More precisely, we construct under this approach a dynamic life table with mortality data of the Canadian province of Quebec. With this table, based on data of a large population, we combine the RRQ mortality data. This results in a dynamic mortality table that can be used by the RRQ to infer the total cost associated with the RRQ pension plan. We mention that the expression “prospective mortality table” also is used for dynamic mortality table. In our paper we will use the latter expression.

In Section 2 we present the notation and assumptions used throughout the paper. We then introduce various approaches for the forecast of the mortality of the population of Quebec in Section 3. We consider a nonparametric method together with two parametric models, namely, the Poisson log-bilinear and the binomial models. We pursue the projection of the time trend and explain the construction of prospective mortality tables from the parametric models. In Section 4 we combine with a Poisson regression model the RRQ mortality data to a dynamic mortality table obtained with the mortality data of the population of Quebec to create a dynamic mortality table that can be used by the RRQ to make any assessment of its total costs. Numerical examples to illustrate the impact of mortality improvements on a pension plan are provided in Section 5.

2. Notation and Assumptions

2.1 Notation

We analyze the changes in mortality as a function of both integer age \( x \) and integer time \( t \). Henceforth,

- \( T_x(t) \) is the remaining life time of an \( x \)-aged individual in calendar year \( t \); this individual will die at age \( x + T_x(t) \) in year \( t + T_x(t) \);
- \( q_x(t) \) is the probability that an \( x \)-aged individual dies in calendar year \( t \) (or dies before age \( x + 1 \)), that is, \( q_x(t) = \Pr[T_x(t) \leq 1] \);
- \( p_x(t) = 1 - q_x(t) \) is the probability that an \( x \)-aged individual in calendar year \( t \) reaches age \( x + 1 \), that is, \( p_x(t) = \Pr[T_x(t) > 1] \);
- \( \mu_x(t) \) is the probability that an \( x \)-aged individual in calendar year \( t \) reaches age \( x + k \), that is, \( \mu_x = \Pr[T_x(t) \geq 1] \);
- \( e_x(t) \) is the expected remaining lifetime of an individual aged \( x \) in year \( t \);
• $\bar{a}_x(t)$ is the pure premium of a life annuity sold to an $x$-year-old individual in year $t$, assuming that the payments are made at the beginning of the year;
• $ETR_{xt}$ is the exposure to risk at age $x$ during year $t$, that is, the total time lived by people aged $x$ in year $t$;
• $D_{xt}$ is the number of deaths recorded at age $x$ during year $t$, from an exposure to risk $ETR_{xt}$;
• $L_{xt}$ is the number of individuals aged $x$ on January 1 of year $t$.

2.2 Assumption of Piecewise Constantness for Forces of Mortality
In this paper we assume that the age-specific forces of mortality are constant within bands of age and time, but allowed to vary from one band to the next. Specifically, given any integer age $x$ and calendar year $t$, it is supposed that
\[
\mu_{x+\xi}(t + \tau) = \mu_x(t), \quad \text{for } 0 \leq \xi, \tau < 1. \tag{2.1}
\]
This is best illustrated with the aid of a coordinate system that has calendar time as abscissa and age as coordinate (called a Lexis diagram after the German demographer who introduced it). Both time scales are divided into yearly bands, which partition the Lexis plane into squared segments. Model (2.1) assumes that the force of mortality is constant within each square, but allows it to vary between squares.

3. Mortality Forecast for the Population Data of Quebec

3.1 Description of the Data
In our study we use the population data of the Canadian province of Quebec, as is provided by the Régie des Rentes du Québec and Statistics Canada. These data are appropriate for the purpose of our study. We have observations for the integer ages $x = x_1, \ldots, x_n$ and $t = t_1, \ldots, t_m$, where $x_1 = 60$, $x_n = 89$, $t_1 = 1971$, and $t_m = 1999$.

3.2 Time Trends
Under a nonparametric approach, the age-specific forces of mortality are estimated with
\[
\hat{\mu}_x(t) = \frac{D_{xt}}{ETR_{xt}},
\]
for $x = x_1, \ldots, x_n$ and $t = t_1, \ldots, t_m$. Figure 1 depicts the evolution of the nonparametric estimates of the forces of mortality over time for some selected ages, separately for men and women. It is clear that the forces of mortality tend to diminish with time. The mortality surfaces for men and women are displayed in Figure 2.

The nonparametric estimates of the age-specific forces of mortality help us to illustrate the evolution of the mortality overtime. However, we need to model the forces of mortality to produce forecasts for the future. We consider two methodologies: the first is based on a Poisson log-bilinear specification, and the second uses a binomial-Gumbel bilinear specification.

3.3 Poisson Log-Bilinear Methodology
Lee and Carter (1992) proposed a simple model for describing the secular change in mortality as a function of a single time index. The method describes the log of a time series of age-specific death rates as the sum of an age-specific component that is independent of time and another component that is the product of a time-varying parameter reflecting the general level of mortality and an age-specific component that represent show rapidly or slowly mortality at each age varies when the general level of mortality changes.

The approach of Brouhns, Denuit, and Vermunt (2002a) consists of substituting Poisson random variation for the number of deaths for an additive error term on the logarithm of mortality rates. It is
worth mentioning that the Poisson distribution is well suited to mortality analyses; see, for example, McDonald (1996a,b,c) for more details. More specifically, we consider that

\[
D_xt \sim \text{Poisson}\left(\frac{\text{ETR}_{xt}\mu_x(t)}{t}\right) \quad \text{with} \quad \mu_x(t) = \exp\left(\alpha_x + \beta_x \kappa_t\right),
\]

for \( x = x_1, \ldots, x_n \) and \( t = t_1, \ldots, t_m \). To ensure identifiability, the parameters are subjected to the constraints

\[
\sum_t \kappa_t = 0 \quad \text{and} \quad \sum_x \beta_x = 1.
\]

Note that the force of mortality has the same log-bilinear form \( \ln \mu_x(t) = \alpha_x + \beta_x \kappa_t \) as in the Lee-Carter model. Interpretation of the parameters is quite simple:

\( \alpha_x \): \( \exp \alpha_x \) reflects the general shape of the mortality schedule \( (x = x_1, \ldots, x_n) \).
\( \beta_x \): represents the age-specific patterns of mortality change \( (x = x_1, \ldots, x_n) \). It indicates the sensitivity of the logarithm of the force of mortality at age \( x \) to variations in the time index \( \kappa_t \). In principle, \( \beta_x \) could be negative at some ages \( x \), indicating that mortality at those ages tends to rise when falling at other ages.
κ_i: represents the time trend (t = t_1, \ldots, t_m). The actual forces of mortality change according to an overall mortality index κ_i modulated by an age response β_x. The shape of the β_x profile tells which rates decline rapidly and which decline slowly over time in response of change in κ_i. When κ_i is linear in time, mortality at each age changes at its own constant exponential rate; assume κ_i = c - κ_0 t, then

$$\frac{d}{dt} \ln \{\mu_x(t)\} = \beta_x \frac{d}{dt} \kappa_i = -\beta_x \kappa_0.$$ 

To estimate the parameters α = (α_{x1}, \ldots, α_{xm}), β = (β_{x1}, \ldots, β_{xm}, and κ = (κ_{x1}, \ldots, κ_{xm}), the model (3.1) is fitted to a matrix of age-specific observed forces of mortality using the maximum likelihood principle, that is, by maximizing the log-likelihood $L(\alpha, \beta, \kappa)$ based on model (3.1). Let us denote as

$$\hat{D}_{xt} = E[D_{xt}] = ETR_{xt} \exp(\alpha_x + \beta_x \kappa_0)$$

the expected number of deaths at age x during year t. Then,

$$L(\alpha, \beta, \kappa) = \ln \left\{ \prod_t \prod_x \left( \frac{\hat{D}_{xt}^\beta \exp(-\hat{D}_{xt})}{D_{xt}} \right) \right\}$$

$$= \sum_t \sum_x \{D_{xt} \ln \hat{D}_{xt} - \hat{D}_{xt} - \ln \{D_{xt}!\}\}$$

$$= \sum_t \sum_x \{D_{xt}(\alpha_x + \beta_x \kappa_0) - ETR_{xt} \exp(\alpha_x + \beta_x \kappa_0)\} + \text{constant}.$$ 

Because of the presence of the bilinear term β_xk_i, it is not possible to estimate the proposed model with commercial statistical packages that implement Poisson regression. Goodman (1979) proposed
an iterative procedure to obtain the MLEs: in iteration step \( \nu + 1 \), a single set of parameters is updated fixing the other parameters at their current estimates using the updating scheme

\[
\hat{\theta}^{(\nu+1)} = \hat{\theta}^{(\nu)} - \frac{\delta L^{(\nu)}}{\delta^2 L^{(\nu)}} / \partial \theta^2,
\]

where \( L^{(\nu)} = L^{(\nu)}(\hat{\theta}^{(\nu)}) \).

In our application there are three sets of parameters, the vectors \( \alpha = (\alpha_{x_1}, \ldots, \alpha_{x_n}) \), \( \beta = (\beta_{x_1}, \ldots, \beta_{x_n}) \), and \( \kappa = (\kappa_{t_1}, \ldots, \kappa_{t_m}) \). The updating scheme is as follows: starting with \( \hat{\alpha}^{(0)}_x = 0 \), \( \hat{\beta}^{(0)}_x = 1 \), and \( \hat{\kappa}^{(0)}_t = 0 \) (random values can also be used), we iterate using the following formulas:

\[
\hat{\alpha}^{(v+1)}_x = \hat{\alpha}^{(v)}_x - \sum_t \frac{(D_{xt} - \hat{D}_{xt})}{\hat{\beta}^{(v)}_x}, \quad \hat{\beta}^{(v+1)}_x = \hat{\beta}^{(v)}_x, \quad \hat{\kappa}^{(v+1)}_t = \hat{\kappa}^{(v)}_t,
\]

\[
\hat{\kappa}^{(v+2)}_t = \hat{\kappa}^{(v+1)}_t - \sum_x \frac{(D_{xt} - \hat{D}_{xt}^{(v+1)}) \hat{\beta}^{(v+1)}_x}{\hat{\kappa}^{(v+1)}_t}, \quad \hat{\alpha}^{(v+2)}_x = \hat{\alpha}^{(v+1)}_x, \quad \hat{\beta}^{(v+2)}_x = \hat{\beta}^{(v+1)}_x,
\]

\[
\hat{\beta}^{(v+3)}_x = \hat{\beta}^{(v+2)}_x - \sum_t \frac{(D_{xt} - \hat{D}_{xt}^{(v+2)}) \hat{\kappa}^{(v+2)}_t}{\hat{\beta}^{(v+2)}_x}, \quad \hat{\alpha}^{(v+3)}_x = \hat{\alpha}^{(v+2)}_x, \quad \hat{\kappa}^{(v+3)}_t = \hat{\kappa}^{(v+2)}_t,
\]

where \( \hat{D}_{xt} = \text{ETR}_{xt} \exp(\hat{\alpha}^{(x)}_x + \hat{\beta}^{(x)}_x \hat{\kappa}^{(x)}_t) \), or the estimated number of deaths after iteration step \( v \). The criterion used to stop the procedure is while the relative increase of the log-likelihood function is smaller than a small fixed number \( \varepsilon \) (say, \( \varepsilon = 0.000001 \)). Let us mention that it is also possible to optimize the Poisson likelihood by monitoring the associated deviance, as described in Renshaw and Haberman (2003b).

The ML estimations of the parameters have to be adapted to fulfill the Lee-Carter constraints; specifically, we switch from \( (\hat{\alpha}, \hat{\beta}, \hat{\kappa}) \) to \( (\alpha^*, \beta^*, \kappa^*) \) given by

\[
\kappa^*_t = (\hat{\kappa}_t - \bar{\kappa}) \sum_x \hat{\beta}_x, \quad \beta^*_x = \sum_x \hat{\beta}_x, \quad \text{and} \quad \alpha^*_x = \hat{\alpha}_x + \hat{\beta}_x \bar{\kappa}.
\]

The new estimates \( \alpha^*, \beta^*, \) and \( \kappa^* \) fulfill the constraints and provide the same \( \hat{D}_{xt} \) since \( \hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t = \alpha^*_x + \beta^*_x \kappa^*_t \). Henceforth, we denote as \( \hat{\alpha}_x, \hat{\beta}_x, \) and \( \hat{\kappa}_t \) the estimated parameters satisfying (3.3).

Note that differentiating the log-likelihood with respect to \( \alpha_x \) gives the equation

\[
\sum_t D_{xt} = \sum_t \hat{D}_{xt} = \sum_t \text{ETR}_{xt} \exp(\hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t).
\]

So the estimated \( \kappa_t \)'s are such that the resulting death rates applied to the actual risk exposure produce the total number of deaths actually observed in the data for each age \( x \). Sizable discrepancies between predicted and actual deaths thus are avoided.

We apply the Poisson modeling to the population data of the province of Quebec. The Poisson parameters \( \alpha_x, \beta_x, \) and \( \kappa \) involved in (3.1) are estimated via maximum likelihood. Figure 3 plots the estimated values of \( \alpha = (\alpha_{x_1}, \ldots, \alpha_{x_n}) \), \( \beta = (\beta_{x_1}, \ldots, \beta_{x_n}) \), and \( \kappa = (\kappa_{t_1}, \ldots, \kappa_{t_m}) \).
Figure 3
α, β, and κ Estimates for Poisson Model
Since we work in a regression framework, it is essential to inspect the residuals. With Poisson random components, deviance residuals

$$\sqrt{2} \times \text{sign}(D_{xt} - \hat{D}_{xt}) \sqrt{\frac{D_{xt}}{\hat{D}_{xt}}} \ln \frac{D_{xt}}{\hat{D}_{xt}} - (D_{xt} - \hat{D}_{xt})$$

are appropriate to monitor the quality of the fit. Figure 4 displays the evolution of residuals through time at different ages. The absence of structure supports the model.

### 3.4 Binomial-Gumbel Bilinear Methodology

The Poisson modeling for $D_{xt}$ can be seen as an approximation of the “true” binomial process generating death counts. The number of deaths $D_{xt}$ at age $x$ during year $t$ has a binomial distribution with parameters $L_{xt}$ and $q_x(t)$. The specification for $\mu_x(t)$ gives

**Figure 4**

**Deviance Residuals at Different Ages for Poisson Model**
with commercial statistical packages that implement Binomial regression. Therefore, we resort as in

\[ D_{xt} \sim \text{Binomial} \left( L_{xt}, q_x(t) \right) \]

Therefore, we resort as in (3.5) for the parameters \( \alpha = (\alpha_{x_1}, \ldots, \alpha_{x_n}) \), \( \beta = (\beta_{x_1}, \ldots, \beta_{x_m}) \), and \( \kappa = (\kappa_{t_1}, \ldots, \kappa_{t_n}) \). Assuming independence between the number of deaths for each combination of age and calendar year, the likelihood for the entire data is the corresponding product of binomial probability factors. The log-likelihood then is given by

\[
L(\alpha, \beta, \kappa) = \ln \left\{ \prod_t \prod_x \left( \frac{L_{xt}}{\hat{q}_{xt}} \right) \left( 1 - \hat{q}_{xt} \right)^{d_{xt}} \right\}
\]

\[
= \sum_t \sum_x (d_{xt} \ln (1 - \hat{q}_{xt}) + d_{xt} \ln \hat{q}_{xt}) + \text{constant}.
\]

Because of the presence of the bilinear term \( \beta_{x_k} \), it is not possible to estimate the proposed model with commercial statistical packages that implement Binomial regression. Therefore, we resort as in the Poisson case to the procedure suggested by Goodman (1979). Denoting

\[
\mu_{xt}^{(v)} = \exp(\hat{\alpha}_x + \hat{\beta}_x(t)) \quad \text{and} \quad q_{xt}^{(v)} = 1 - \exp(-\mu_{xt}^{(v)}) = 1 - p_{xt}^{(v)},
\]

the iterative procedure yielding the MLEs can be written as follows:

\[
\hat{\alpha}_x^{(v+1)} = \hat{\alpha}_x^{(v)} - \frac{\sum_{t=t_{\min}}^{t_{\max}} \mu_{xt}^{(v)} \left( D_{xt} \frac{q_{xt}^{(v)}}{q_{xt}^{(v+1)}} - L_{xt} \right)}{\sum_{t=t_{\min}}^{t_{\max}} \mu_{xt}^{(v)} \left( D_{xt} \frac{q_{xt}^{(v+1)}}{q_{xt}^{(v+1)}} - L_{xt} \right)},
\]

\[
\hat{\beta}_x^{(v+1)} = \hat{\beta}_x^{(v)}, \quad \hat{\kappa}_t^{(v+1)} = \hat{\kappa}_t^{(v)}.
\]

\[
\hat{\alpha}_x^{(v+2)} = \hat{\alpha}_x^{(v+1)}, \quad \hat{\kappa}_t^{(v+2)} = \hat{\kappa}_t^{(v+1)}.
\]

\[
\hat{\beta}_x^{(v+2)} = \hat{\beta}_x^{(v+1)}, \quad \hat{\kappa}_t^{(v+3)} = \hat{\kappa}_t^{(v+2)}.
\]

\[
\hat{\alpha}_x^{(v+3)} = \hat{\alpha}_x^{(v+2)}, \quad \hat{\beta}_x^{(v+3)} = \hat{\beta}_x^{(v+2)}.
\]

We apply the Binomial modeling to the population data of the province of Quebec. The parameters \( \alpha = (\alpha_{x_1}, \ldots, \alpha_{x_n}) \), \( \beta = (\beta_{x_1}, \ldots, \beta_{x_m}) \), and \( \kappa = (\kappa_{t_1}, \ldots, \kappa_{t_n}) \) involved in (3.5) are estimated via maximum likelihood. Figure 5 plots the estimated values of \( \alpha = (\alpha_{x_1}, \ldots, \alpha_{x_n}) \), \( \beta = (\beta_{x_1}, \ldots, \beta_{x_m}) \), and \( \kappa = (\kappa_{t_1}, \ldots, \kappa_{t_n}) \).

To monitor the quality of the fit by the Binomial-Gumbel model, the deviance residuals given by
Figure 5
\(\alpha, \beta, \text{ and } \kappa \) Estimates for Binomial Model
The evolution of residuals through time at different ages are illustrated in Figure 6. As one can see, the absence of structure supports the model.

### 3.5 Selection of the Model

To compare the Poisson and the binomial models, we use the percentages of total deviance explained by each model (i.e., the pseudo $R^2$), which are given in Table 1.

Considering the data at our disposal, a binomial or a Poisson modeling for the number of deaths can be envisaged. Contrarily to the binomial model, the Poisson model explicitly takes into account the

![Figure 6: Deviance Residuals at Different Ages for Binomial Model](image-url)
Table 1
Percentages of Total Deviance by Poisson and Binomial Models

<table>
<thead>
<tr>
<th></th>
<th>Poisson Model</th>
<th>Binomial Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Males</td>
<td>86.30%</td>
<td>86.18%</td>
</tr>
<tr>
<td>Females</td>
<td>87.08</td>
<td>87.00</td>
</tr>
</tbody>
</table>

exposure to risk, which maybe considered as an advantage for the application we have in mind. Moreover, the fit obtained with the Poisson model is slightly better than the binomial fit. For these reasons, we decided to keep the Poisson model for the death counts stratified by age and time.

3.6 Projection of the Time Index
The time factor \( \kappa_t \) is intrinsically viewed as a stochastic process, and Box-Jenkins techniques are then used to estimate and forecast \( \kappa_t \) within an ARIMA \((p, d, q)\) times series model, which takes the general form

\[
\nabla^d \kappa_t = \rho + \Theta_q(B) \frac{\Phi_p(B)}{\sigma^2},
\]

where

\( B \) is the delay operator, \( B(\kappa_t) = \kappa_{t-1}, B^2(\kappa_t) = \kappa_{t-2}, \ldots \),

\( \nabla = 1 - B \) is the difference operator, \( \nabla \kappa_t = \kappa_t - \kappa_{t-1}, \nabla^2 \kappa_t = \kappa_t - 2\kappa_{t-1} + \kappa_{t-2}, \ldots \),

\( \Theta_q(B) \) is the moving average polynomial, with coefficients \( \Theta = (\theta_1, \theta_2, \ldots, \theta_q) \),

\( \Phi_p(B) \) is the autoregressive polynomial, with coefficients \( \phi = (\phi_1, \phi_2, \ldots, \phi_p) \), and \( \varepsilon_t \) is white noise with variance \( \sigma^2 \).

The parameters of the models are \( \mu, \theta, \phi, \) and \( \sigma^2 \). The method used to obtain estimates for the ARIMA parameters is conditional least squares. The choice of the optimal model has been based on the AIV values. This yielded to select an ARIMA \((1, 1, 0)\) for males and an ARIMA \((0, 1, 1)\) for females. Forecast values of time parameters \( \kappa_t \) (here \( t \) is beyond the observation period) can be seen in Figure 7.

As is discussed in the next sections, the parameter estimates of the Poisson model and the \( \kappa_t \) forecasts can be used to obtain projected age-specific mortality rates, life expectancies, and annuity single premiums. Projected deaths rates at some selected ages are displayed in Figure 8.

3.7 Completion of the Life Tables
Data at old ages produce suspect results (because of small risk exposures): the pattern at old and very old ages is heavily affected by random fluctuations because of their scarcity. Sometimes data above some high age are not available at all. Recently some in-depth demographic studies provided a sound knowledge about the slope of the mortality curve at very old ages. It has been documented that the force of mortality is slowly increasing at very old ages, approaching a rather flat shape. The deceleration of the rate of mortality increase can be explained by the selective survival of healthier individuals to older ages (see, e.g., Horiuchi and Wilmoth 1998 for more details).

Demographers and actuaries suggested various techniques to complete forces of mortality at old ages, among others the influential work by Lindbergson (2001), Coale and Guo (1989), Coale and Kisker (1990), Thatcher (1999), Thatcher, Kannisto, and Vaupel (1998), and Thatcher, Kannisto, and Andrew (2002). See also a more extensive list of references in Delwarde and Denuit (2005). We refer the interested reader to Buettner (2002) for an interesting discussion. The reader may also consult the Kannisto-Thatcher Database on Old Age Mortality (http://www.demogr.mpg.de/). In this paper
Figure 7
Forecasting of Time Parameters $\kappa_t$

Figure 8
Projected Death Rates at Different Ages
we use a simple and powerful method proposed by Denuit and Goderniaux (2004). This method is briefly presented below.

The starting point is standard: a constrained log-quadratic regression model of the form

$$\ln q(t) = a + bx + cx^2 + \varepsilon_{xt}$$

with $\varepsilon_{xt}$ i.i.d. $N(0, \sigma^2)$ fitted separately to each calendar year $t(t = t_1, \ldots, t_m)$ and to ages $(x)$ 75 and over. The difference lies in the constraints we impose. First, we fix a closure constraint

$$q_{130}(t) = 1 \quad \text{for all } t.$$  

Even if the human life span shows no sign of approaching a fixed limit imposed by biology or other factors (see, e.g., Wilmoth 1997 or Wilmoth et al. 2000), it seems reasonable to retain as a working assumption that the limit age 130 will not be exceeded. Second, we assume an inflexion constraint

$$\frac{\partial q_x(t)}{\partial x} \bigg|_{x=130} = 0 \quad \text{for all } t.$$  

These two constraints yield the following relation between the $a_t$’s, $b_t$’s, and $c_t$’s for each calendar time $t$:

$$a_t + b_t x + c_t x^2 = c_t (130 - x)^2,$$  

for $x = 75, 76, \ldots, x_n$ and $t = t_1, \ldots, t_m$. The $c_t$’s then are estimated on the basis of the series \(\{q_x(t), x = 75, 76, \ldots, x_n\}\) relating to year $t$. It is worth mentioning that the two requirements underlying the modeling of the $q_x(t)$ for high $x$ are in line with the empirical demographic evidence.

The completed data set then is obtained as follows. We keep the original $\hat{q}_x(t)$ for $x = x_1, \ldots, 85$, and we replace the death probabilities for ages over 85 with the fitted values coming from the constrained quadratic regression. The results for calendar years 1975, 1985, and 1995 can be seen in Figure 9. This furnishes a rectangular array of data displayed in Figure 10.

4. APPLICATION TO A PENSION PLAN

We are now interested in the evaluation of the total cost associated to a given pension plan (private or public). More precisely, we want to base this evaluation on a dynamic mortality table obtained from the mortality experience of this pension plan and its characteristics. Often the pension plan is not large enough to base the construction of a dynamic mortality table solely on its own experience. To obtain such a table, one can combine a dynamic mortality table built from a larger population and the mortality and the characteristics of the population of the pension plan. In the application that follows, we use a Poisson regression model to combine the mortality data of the Régie des Rentes du Québec pension plan to a dynamic mortality table, which is obtained from the mortality data of the whole province of Quebec. The resulting dynamic mortality table can be used by the RRQ to make any assessment of the total cost associated with the plan. This approach also could be used for smaller private pension plans using any appropriate basic dynamic mortality table.

4.1 Description of the RRQ and Data

The Régie des Rentes du Québec is a provincial public pension plan that came into effect in January 1966 at the same time as its federal substitute, the Canadian Pension Plan (CPP). The goal of the RRQ is mainly to provide to all members of the paid labor force in the province of Quebec a minimum retirement income and benefits in the event of disability or death before retirement. All workers in the province must contribute with his or her employer to the funding via monthly contributions, which are determined by the provincial government. Annuities provided by the RRQ depend on the worker’s wage and are indexed each year to compensate for inflation. Disability annuities are made up of two parts, a uniform rate and an additional benefit for dependent children. Note that a disabled individual will
Figure 9
Completion of Death Probabilities for Years 1975, 1985, and 1995
see his or her disability annuity transformed into a retirement annuity on their sixty-fifth birthday. Retirement annuities are strictly proportional to the average wage of the retired annuitant. They are set in order to provide 25% of the annuitant’s income before retirement up to 25% of the average national wage.

Let us briefly describe the data file provided by the RRQ for the analysis that follows. The data file contains the individual data (1,606,141 records) for all members from the inception of the RRQ to 2003. In addition to the year of birth and of death (if applicable), we use as covariates whether the individual retired before 65 (early retirement) or after this age, whether he or she is disabled or not at retirement, the amount of his or her annuity (expressed as a percentage of the maximal amount, corresponding to full service and maximum pensionable earnings), and whether he or she benefited from payments made by the CPP for individuals who spent part of their career in Canada, but outside Quebec.

4.2 Descriptive Analysis
Let us first compare the mortality experienced by the RRQ annuitants to the whole Quebec population. Figure 11 displays the ratio of the mortality rates for the RRQ and the entire Quebec population. It can be seen that the situation depends on gender. For males, the ratio is around 1 and above, showing no significant reduction of mortality for the RRQ annuitants. For females, however, Figure 11 suggests a better mortality for the RRQ annuitants.

4.3 Impact of RRQ Annuitant Characteristics
Let us start with a description of the RRQ database. We see from the histograms displayed in Figures 12 to 14 that

- The vast majority of the RRQ annuitants do not benefit from a pension paid by the CPP;
- A majority of individuals retired before age 65; and
• Men benefit from higher pensions than women: modal class for men is 100, i.e., a percentage of 100%, whereas modal class for women is 30–50, corresponding to a percentage ≥ 30% and < 50%.

Let us now study the possible marginal effect of the explanatory variables on the mortality experienced by the RRQ annuitants. This is done in Figures 15 to 19, where the mortality rates by age and gender are represented separately for the subpopulations indexed by the explanatory variables. We can see there that
Early retirement (i.e., before age 65) leads to better mortality rates;
Receiving payments from both RRQ and CPP has only a weak impact on the mortality rates;
The mortality rates increase for individuals who are disabled at retirement;
The mortality rates globally seem to decrease with the percentage for men, but the situation is somewhat ambiguous for women.

The next section proposes a Poisson regression model for death counts incorporating exogeneous information.

4.4 Poisson Regression for Annual RRQ Death Counts
Let us now explain the annual death counts observed for the RRQ annuitants, as a function of the four explanatory variables we have at our disposal. More precisely, we fit a different Poisson regression model for men and women, with the annual death counts $D_{xt}^{\text{RRQ}}$ for RRQ members aged $x$ in year $t$ as

$$D_{xt}^{\text{RRQ}} \sim \text{Poisson} \left( \text{ETR}_{xt}^{\text{RRQ}} \mu_x(t) \exp \left( \beta_0 + \sum_{j=1}^{s} \beta_j z_{xjt} \right) \right),$$

where $\text{ETR}_{xt}^{\text{RRQ}}$ is the exposure to risk for RRQ at age $x$ in year $t$, $\mu_x(t)$ is the fitted mortality rate for the Quebec population (obtained in the preceding section), and the $z_{xjt}$'s are binary variables coding the four categorical explanatory variables at our disposal. The number $s + 1$ of parameters depends on the sex ($s = 9$ for men and $s = 8$ for women).

For men, we group the percentages of maximal pension amounts in seven classes 0–10, 10–30, 30–50, 50–70, 70–85, 85–100, and 100. For women, the classes are 0–10, 10–30, 30–50, 70–100, and 100. We also indicate if an individual has retired before ($< 65$) or, given that he or she has retired at or after ($\geq 65$) age 65, whether he or she is disabled or not. Finally, we take into consideration whether the annuity is fully paid by the RRQ or jointly with the CPP. We use the logarithm of the product of the exposure to risk with the Quebec mortality rate as offset. As an example, for men who have retired at age 63 and received a pension annuity between 70% and 85% of the maximal pension amount, the
Figure 13
RRQ Annuitant Characteristics: Early Retirement and Disability

covariates are coded as follows: $z_{it1} = 0$, $z_{it2} = 0$, $z_{it3} = 0$, $z_{it4} = 1$, $z_{it5} = 0$, $z_{it6} = 0$, $z_{it7} = 0$, $z_{it8} = 0$, and $z_{it9} = 0$.

After having taken into consideration all the elements just mentioned, we have obtained the two models displayed in Tables 2 and 3. For both models, all the regression coefficients significantly differ from 0.

Boxplots of the residuals are displayed in Figure 20. No structure emerges from these figures, so that the data seem to agree with the model.
As a result, the forces of mortality for the annuitants of the RRQ correspond to the forces of mortality estimated for the population of Quebec multiplied by the factors displayed in Tables A.1 and A.2, given in the Appendix.

5. **Numerical Examples**

In this section we will apply prospective mortality tables and the Poisson regression model discussed in the preceding sections in particular to the RRQ data. More precisely, we analyze in three different
Figure 15
Marginal Effects of the Explanatory Variables on RRQ Mortality Experience: People Retired after and before Age 65
Figure 16
Marginal Effects of Explanatory Variables on RRQ Mortality Experience: People Retired after Age 65 and Not Disabled
Figure 17
Marginal Effects of Explanatory Variables on RRQ Mortality Experience: People Retired after Age 65 and Disabled
Figure 18
Marginal Effects of Explanatory Variables on RRQ Mortality Experience: Canadian Financing
Figure 19
Marginal Effects of Explanatory Variables on RRQ Mortality Experience: Amount of Annuity
examples the impact of mortality improvement on the expected residual lifetime, on annuity prices, and on the solvency of a general/supplemental pension plan.

### 5.1 Impact of Mortality Improvement on the Expected Residual Lifetime

We begin our analysis with a look at the temporal evolution of the expected residual lifetime given by

\[
E[K_x(t)] = e_x(t) = \sum_{k=0}^{\omega-x-1} (\nu_p)(t)
\]

\[= \sum_{k=0}^{\omega-x-1} \left( \prod_{i=0}^{k-1} p_{x+i}(t+i) \right),
\]

where \(K_x(t)\) is the curtate remaining lifetime corresponding to \(T_x(t)\), that is, the integer part of \(T_x(t)\).

We consider three categories of the RRQ pension plan for males and females, namely, the category with the worst (category A) and the best (category B) observed mortality and one that mostly represents a Quebec pensioner. This last category (category C) regroups nondisabled pensioners who have always worked in the province of Quebec. Also, these pensioners have all left the workforce after the age of 65 and always gained an income higher than the maximum annual earnings on which contributions can be made to the Quebec Pension Plan (Can $40,500 in 2004). The details associated with these categories are given in Tables 4 and 5.

Note that the effect column in Tables 4 and 5 refers to the total multiplicative impact of retiree characteristics on the force of mortality, and the proportion column gives the percentage of pensioners observed in each category for the last year of the RRQ data file.

### Table 2

**Parameters of the Poisson Regression Model for RRQ Annuitants—Males**

<table>
<thead>
<tr>
<th>(j)</th>
<th>Covariates (z_{xij})</th>
<th>Estimate of (\beta_j)</th>
<th>Std Error</th>
<th>(\chi^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Intercept</td>
<td>0.1463</td>
<td>0.0080</td>
<td>338.00</td>
</tr>
<tr>
<td>1</td>
<td>10≤%Max&lt;30</td>
<td>-0.1086</td>
<td>0.0088</td>
<td>153.09</td>
</tr>
<tr>
<td>2</td>
<td>30≤%Max&lt;50</td>
<td>-0.2939</td>
<td>0.0085</td>
<td>572.48</td>
</tr>
<tr>
<td>3</td>
<td>50≤%Max&lt;70</td>
<td>-0.2577</td>
<td>0.0083</td>
<td>967.06</td>
</tr>
<tr>
<td>4</td>
<td>70≤%Max&lt;85</td>
<td>-0.2905</td>
<td>0.0082</td>
<td>1258.48</td>
</tr>
<tr>
<td>5</td>
<td>85≤%Max&lt;100</td>
<td>-0.3655</td>
<td>0.0076</td>
<td>2296.91</td>
</tr>
<tr>
<td>6</td>
<td>Age≥65 &amp; disability = No</td>
<td>0.0370</td>
<td>0.0039</td>
<td>89.06</td>
</tr>
<tr>
<td>7</td>
<td>Age≥65 &amp; disability = Yes</td>
<td>-0.2380</td>
<td>0.0078</td>
<td>173.02</td>
</tr>
<tr>
<td>8</td>
<td>Pro. + Fed. (RRQ + CPP)</td>
<td>0.0565</td>
<td>0.0070</td>
<td>65.80</td>
</tr>
</tbody>
</table>

### Table 3

**Parameters of the Poisson Regression Model for RRQ Annuitants—Females**

<table>
<thead>
<tr>
<th>(j)</th>
<th>Covariates (z_{xij})</th>
<th>Estimate of (\beta_j)</th>
<th>Std Error</th>
<th>(\chi^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Intercept</td>
<td>0.0565</td>
<td>0.0070</td>
<td>65.80</td>
</tr>
<tr>
<td>1</td>
<td>10≤%Max&lt;30</td>
<td>-0.0565</td>
<td>0.0078</td>
<td>173.02</td>
</tr>
<tr>
<td>2</td>
<td>30≤%Max&lt;50</td>
<td>-0.1028</td>
<td>0.0084</td>
<td>330.96</td>
</tr>
<tr>
<td>3</td>
<td>50≤%Max&lt;70</td>
<td>-0.1525</td>
<td>0.0077</td>
<td>577.50</td>
</tr>
<tr>
<td>4</td>
<td>70≤%Max&lt;85</td>
<td>-0.1852</td>
<td>0.0089</td>
<td>783.02</td>
</tr>
<tr>
<td>5</td>
<td>85≤%Max&lt;100</td>
<td>-0.2498</td>
<td>0.0325</td>
<td>54.64</td>
</tr>
<tr>
<td>6</td>
<td>Age≥65 &amp; disability = No</td>
<td>0.0720</td>
<td>0.0058</td>
<td>152.01</td>
</tr>
<tr>
<td>7</td>
<td>Age≥65 &amp; disability = Yes</td>
<td>0.7046</td>
<td>0.0122</td>
<td>3323.59</td>
</tr>
<tr>
<td>8</td>
<td>Pro. + Fed. (RRQ + CPP)</td>
<td>-0.0440</td>
<td>0.0165</td>
<td>7.08</td>
</tr>
</tbody>
</table>
Figure 20

Boxplots of Deviance Residuals

The results of Figure 21 lead to similar conclusions as the ones made with the nonparametric approach performed previously. Mainly we observe that the mortality of the females is, in each category, better than for the males and that the evolution in time of the mortality at a given age differs. We have noticed that the improvement of the mean residual life is mostly significant between the ages 65 and 85 and becomes less significant at higher ages. This leads, in time, to pensioners living longer lives, that is, reaching an age closer to the limit age without, however, an increase in the limit age.

We also observe significant differences in the expected residual lifetime of the pensioners in the different categories, which confirms the relevance of the Poisson regression model to discriminating the retirees. For example, the expected residual lifetime at age 65 in 2005 for the males of categories A, B, and C are, respectively, −33%, 13%, and 5% higher than the ones obtained with the GAM 83 table and for the females −14%, 25%, and 18%. In 2015 these percentages become −28%, 18%, and 11% for the males and −7%, 31%, and 24% for the females. The increase in time of the percentages confirms our previous observation of a better mortality in time.

5.2 Impact of Mortality Improvement on Annuity Prices

In our second example we look at the impact of the temporal evolution of the mortality on the price of life annuities given by

<table>
<thead>
<tr>
<th>Name</th>
<th>Percentage</th>
<th>Financing</th>
<th>AgeRtr</th>
<th>Disability</th>
<th>Effect</th>
<th>Proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>M-A</td>
<td>0≤%Max&lt;10</td>
<td>RRQ</td>
<td>≥65</td>
<td>Yes</td>
<td>2.3147</td>
<td>0.015%</td>
</tr>
<tr>
<td>M-B</td>
<td>85≤%Max&lt;100</td>
<td>RRQ+CPP</td>
<td>&lt;65</td>
<td>No</td>
<td>0.7677</td>
<td>0.875</td>
</tr>
<tr>
<td>M-C</td>
<td>%Max=100</td>
<td>RRQ</td>
<td>&lt;65</td>
<td>No</td>
<td>0.9124</td>
<td>12.687</td>
</tr>
</tbody>
</table>
Table 5
Details Associated with Categories for Females in the RRQ Pension Plan

<table>
<thead>
<tr>
<th>Name</th>
<th>Percentage</th>
<th>Financing</th>
<th>AgeRtr</th>
<th>Disability</th>
<th>Effect</th>
<th>Proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>F-A</td>
<td>0≤%Max&lt;10</td>
<td>RRQ</td>
<td>≤65</td>
<td>Yes</td>
<td>1.9119</td>
<td>0.036%</td>
</tr>
<tr>
<td>F-B</td>
<td>85≤%Max&lt;100</td>
<td>RRQ+CPP</td>
<td>&lt;65</td>
<td>No</td>
<td>0.7045</td>
<td>0.159</td>
</tr>
<tr>
<td>F-C</td>
<td>10≤%Max&lt;30</td>
<td>RRQ</td>
<td>&lt;65</td>
<td>No</td>
<td>0.8527</td>
<td>8.702</td>
</tr>
</tbody>
</table>

\[
\bar{a}_x(t) = \sum_{k=0}^{\omega-x-1} e^k \left( i p_x(t) \right)
\]

\[
= \sum_{k=0}^{\omega-x-1} e^k \left( \prod_{i=0}^{k-1} p_{x+i}(t + i) \right).
\]

We choose an interest rate of 6%, which is comparable to the observed yield to maturity on long-term bonds. Figure 22 illustrates the evolution in time of annuity prices for the categories of the RRQ data defined in the previous example.

Not surprisingly, the results corroborate the ones obtained for the evolution of the mean residual lifetime given that it also can be seen as a life annuity with a zero interest rate.

Figure 22 allows us to see the effect of mortality improvements on annuity prices, that is, that its improvement is attenuated by an increase in the interest rate. As was done in the first example, we can look at the differences in the price of annuities obtained with a dynamic mortality table and the GAM 83 mortality table (computed with an interest rate of 6%). For example, an annuity issued at 65 in 2005 for the three male categories is –24%, 6%, and 2% higher than the price obtained with the GAM 83. In 2015 these percentages, respectively, become –20%, 9%, and 5%. As for the females in the same categories, the increases in percentage are –9%, 11%, and 8% for 2005 and –5%, 14%, and 11% for 2015.
5.3 Impact of Mortality Improvement on the Solvency of Pension Plans

We now analyze the impact of mortality improvements in time on the solvency of general/supplemental pension plans.

We consider a pension plan with \( n = 1000 \) members divided into 78 categories with the same proportions as those observed in the RRQ pension plan between years 1994 and 2003. We assume that at a given evaluation date, the solvency ratio of the pension plan is 100%, which means that the assets are equal to the liabilities that are calculated with a static mortality table (e.g., GAM 83). Such a ratio seems to indicate a good financial situation, but one must remember that the mortality improvements are ignored with such a mortality table. In fact, one should expect more important liabilities with the improvement of mortality displayed in dynamic life tables.

To illustrate such a problem, we compute the ruin probability (by Monte Carlo simulation), which corresponds to the probability that the pension assets go below zero in the future. We assume that the future mortality improvements of the pension members will correspond to those predicted with the dynamic mortality table obtained previously for the RRQ pension plan. At valuation date, we assume that the pension assets and liabilities are equal. For sake of simplicity, we also assume that the assets are invested at a deterministic rate of 4% or 6% and that the annuity payments are made at the beginning of each year. In Table 6 we observe that the ruin probabilities increase as the evaluation is made at later dates.

These results are confirmed by Figure 23, which shows the evolution of the expected annuity payments in the future in several different contexts. The computation of these payments is made with the GAM 83 static mortality table and a dynamic mortality table in which the improvement of the mortality begins in years 2000, 2005, 2010, 2015, or 2020. From this figure we observe that according to the projected tables, the expected annuity payments decrease less rapidly than those computed with the GAM 83 mortality table.
Table 6
Pension Liabilities (with Projected Mortality Tables and GAM 83) and Ruin Probabilities

<table>
<thead>
<tr>
<th>Valuation Date</th>
<th>Pension Liabilities (Projected Tables)</th>
<th>Pension Liabilities (GAM 83)</th>
<th>Ruin Probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( i = 4% )</td>
<td>( i = 6% )</td>
<td>( i = 4% )</td>
</tr>
<tr>
<td>2000</td>
<td>67,659,522</td>
<td>56,753,274</td>
<td>66,231,698</td>
</tr>
<tr>
<td>2005</td>
<td>68,930,972</td>
<td>57,657,921</td>
<td>66,231,698</td>
</tr>
<tr>
<td>2010</td>
<td>70,153,586</td>
<td>58,521,615</td>
<td>66,231,698</td>
</tr>
<tr>
<td>2015</td>
<td>71,333,727</td>
<td>59,350,133</td>
<td>66,231,698</td>
</tr>
<tr>
<td>2020</td>
<td>72,471,688</td>
<td>60,144,082</td>
<td>66,231,698</td>
</tr>
</tbody>
</table>

Figure 23
Projection of Expected Pension Annuity Payments for a Representative RRQ Portfolio of 1000 Annuitants as a Function of Year of Insurance
6. Discussion

In this paper we have disregarded the sampling errors in the Lee-Carter parameters $\alpha_x$, $\beta_x$, and $\kappa_t$, in the ARIMA parameters as well as in the regression parameters in the Cox regression model. These errors obviously are related to the uncertainty inherent to annuity values. Since annuity single premiums are complicated nonlinear functions of the Lee-Carter, ARIMA, and Cox regression parameters, an analytic approach is out of reach, and numerical procedures have to be contemplated. Bootstrap methods are particularly useful in that respect. The key idea behind the bootstrap is to resample from the original data (either directly or via a fitted model) to create replicate data sets, from which the variability of the quantities of interest can be assessed. Because this approach involves repeating the original data analysis procedure with many replicate sets of data, it is sometimes called a computer-intensive method. Bootstrap techniques are particularly useful when, as in our problem, theoretical calculation with the fitted model is too complex.

The three sources of uncertainty that have to be combined are the sampling fluctuation in the $\alpha_x$, $\beta_x$, and $\kappa_t$ parameters, the forecast error in the $\kappa_t$ parameters, and the sampling fluctuation in the Cox regression parameters. In a Poisson regression framework, Brouhns, Denuit, and Vermunt (2002a,b) sampled directly from the approximate multivariate Normal distribution of the maximum likelihood estimators $\hat{\alpha}$, $\hat{\beta}$, $\hat{\kappa}$. Brouhns, Denuit, and Keilegom (2005) sampled from the death counts (under a Poisson error structure). Specifically the bootstrapped death counts are obtained by applying a Poisson noise to the observed numbers of deaths. The bootstrapping procedure also can be achieved in a number of alternative ways. For instance, here we could follow a “generation” and generate pseudo-death counts from a multinomial distribution. Still another possibility is to bootstrap from the residuals of the fitted Poisson log-bilinear model. The deviance residuals should be independent and identically distributed (provided the model is well specified). Therefore, it is possible to reconstitute bootstrapped residuals, and therefrom bootstrapped mortality data. Note that here a double bootstrap procedure is needed, since we face two populations: the general Quebec population and that of the RRQ.

A pragmatic regression approach has been used in this paper to elaborate a projected life table for the RRQ. Basically the large data set (population of Quebec) is used to produce a baseline life table that becomes an explanatory variable in a Cox regression model to explain the mortality in the small data set (RRQ). A credibility approach could have been used instead. In this case the number of deaths recorded among the RRQ members would have been a priori taken Poisson distributed, with a mean equal to the product of the RRQ exposure to risk times the population death rates times a specific Gamma random effect with unit mean (so that the number of deaths is negative binomial). Note that the large data set again is used to produce a baseline life table for the RRQ. But in this case this life table is not used in a Cox regression model but serves as the a priori death rate for the RRQ. Considering the series of the $D^{\text{RRQ}}_{\text{st}}$’s for fixed $x$, assumed to be conditionally independent given the random effect, the a posteriori distribution indicates how population death rates are transformed to fit the RRQ experience.
## APPENDIX

### Table A.1

Multiplicative Factors to Forces of Mortality of the Population of Quebec to Obtain Forces of Mortality of RRQ Annuitants—Males

<table>
<thead>
<tr>
<th>Class No.</th>
<th>Percentage</th>
<th>Financing</th>
<th>AgeRtr</th>
<th>Disability</th>
<th>Multiplicative Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 ≤ % Max &lt; 10</td>
<td>RRQ</td>
<td>&lt; 65</td>
<td>No</td>
<td>1.1575</td>
</tr>
<tr>
<td>2</td>
<td>0 ≤ % Max &lt; 10</td>
<td>RRQ</td>
<td>≥ 65</td>
<td>No</td>
<td>1.2012</td>
</tr>
<tr>
<td>3</td>
<td>0 ≤ % Max &lt; 10</td>
<td>RRQ</td>
<td>≥ 65</td>
<td>Yes</td>
<td>2.3147</td>
</tr>
<tr>
<td>4</td>
<td>0 ≤ % Max &lt; 10</td>
<td>RRQ + CPP</td>
<td>&lt; 65</td>
<td>No</td>
<td>1.1064</td>
</tr>
<tr>
<td>5</td>
<td>0 ≤ % Max &lt; 10</td>
<td>RRQ + CPP</td>
<td>≥ 65</td>
<td>No</td>
<td>1.1581</td>
</tr>
<tr>
<td>6</td>
<td>0 ≤ % Max &lt; 10</td>
<td>RRQ + CPP</td>
<td>≥ 65</td>
<td>Yes</td>
<td>2.2124</td>
</tr>
<tr>
<td>7</td>
<td>10 ≤ % Max &lt; 30</td>
<td>RRQ</td>
<td>&lt; 65</td>
<td>No</td>
<td>1.0384</td>
</tr>
<tr>
<td>8</td>
<td>10 ≤ % Max &lt; 30</td>
<td>RRQ</td>
<td>≥ 65</td>
<td>No</td>
<td>1.0776</td>
</tr>
<tr>
<td>9</td>
<td>10 ≤ % Max &lt; 30</td>
<td>RRQ</td>
<td>≥ 65</td>
<td>Yes</td>
<td>2.0765</td>
</tr>
<tr>
<td>10</td>
<td>10 ≤ % Max &lt; 30</td>
<td>RRQ + CPP</td>
<td>&lt; 65</td>
<td>No</td>
<td>0.9925</td>
</tr>
<tr>
<td>11</td>
<td>10 ≤ % Max &lt; 30</td>
<td>RRQ + CPP</td>
<td>≥ 65</td>
<td>No</td>
<td>1.0299</td>
</tr>
<tr>
<td>12</td>
<td>10 ≤ % Max &lt; 30</td>
<td>RRQ + CPP</td>
<td>≥ 65</td>
<td>Yes</td>
<td>1.9848</td>
</tr>
<tr>
<td>13</td>
<td>30 ≤ % Max &lt; 50</td>
<td>RRQ</td>
<td>&lt; 65</td>
<td>No</td>
<td>0.9440</td>
</tr>
<tr>
<td>14</td>
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### Table A.2
**Multiplicative Factors to Forces of Mortality of the Population of Quebec to Obtain Forces of Mortality of RRQ Annuitants—Females**

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<th>Class No.</th>
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The results and the opinions given in this paper are the sole responsibility of the authors.

### References


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