STOCHASTIC APPROXIMATIONS OF PRESENT VALUE FUNCTIONS

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Abstract

The aim of this paper is to apply the method proposed by Denuit, Genest and Marceau (1999) for deriving stochastic upper and lower bounds on the present value of a sequence of cash flows, where the discounting is performed under a given stochastic return process. The convex approximation provided by Goovaerts, Dhaene and De Schepper (2000) and Goovaerts and Dhaene (1999) is then compared to these stochastic bounds. On the basis of several numerical examples, it will be seen that the convex approximation seems reasonable.

Key words and phrases: Dependence, Stochastic Dominance, Stochastic Annuities

Résumé.


Mots-clé: dépendance, dominance stochastique, annuités stochastiques
1 Introduction

Let $V_t$ be the present value at time 0 of an amount of $\alpha_t$ paid at time $t$. The stochastic discounted value at time 0 of payments of amount $\alpha_t$ made at times $t = 1, 2, \cdots, n$ is then given by

$$Z_n = V_1 + V_2 + \cdots + V_n. \quad (1.1)$$

Consider for instance an insurance company facing payments of amount $\alpha_t$ at times $t = 1, 2, \cdots, n$; the present value of these $n$ deterministic payments is given by (1.1).

The $V_t$'s involved in (1.1) are obviously correlated, so that the convenient independence assumption for the summands in $Z_n$ is not realistic. As a consequence, an exact expression for the cumulative distribution function of $Z_n$ requires the knowledge of the joint distribution of the random vector $(V_1, V_2, \cdots, V_n)$, which is in general not available. Goovaerts, Dhaene and De Schepper (2000) recently proposed to circumvent this problem by approximating $Z_n$ by means of a random variable $\tilde{Z}_n$ dominating the original $Z_n$ in the convex sense. If we denote by $F_1, F_2, \cdots, F_n$ the respective distribution functions of $V_1, V_2, \cdots, V_n$ involved in (1.1), $\tilde{Z}_n$ is given by

$$\tilde{Z}_n = F_1^{-1}(U) + F_2^{-1}(U) + \cdots + F_n^{-1}(U),$$

where $U$ is a unit uniform random variable and the $F_i^{-1}$'s are the quantile functions associated to the $F_i$'s. We obviously have that $EZ_n = E\tilde{Z}_n$ and it can be shown that the inequalities

$$E \max\{Z_n - d, 0\} \leq E \max\{\tilde{Z}_n - d, 0\} \quad (1.2)$$

hold for any $d \geq 0$ (that is, $Z_n$ is smaller than $\tilde{Z}_n$ in the convex order).

Since $\tilde{Z}_n$ precedes $Z_n$ in the convex sense, the approximation $\tilde{Z}_n$ is considered as less favorable by all the risk-averse decision-makers, and the method is thus conservative. Moreover, the cumulative distribution function of $\tilde{Z}_n$ enjoys an explicit expression and is particularly easy to handle. On the basis of numerical illustrations performed in a situation where the exact cumulative distribution function of $Z_n$ can be obtained, Goovaerts et al. (1999) showed that the cumulative distribution functions of $Z_n$ and $\tilde{Z}_n$ seem to be rather close.

The problem of estimating the distribution of $Z_n$ has been studied, among others, by Beekman and Fuelling (1991), De Schepper and Goovaerts (1992), Dufresne (1990), Frees (1990), Parker (1994c,1997), De Schepper, Teunen, Goovaerts (1994) and Vanneste, Goovaerts and Labie (1994). This paper aims to carry on with Goovaerts et al.’s (1999) approach by providing lower and upper bounds on $Z_n$ in the stochastic dominance sense, using the method proposed in Denuit, Genest and Marceau (1999). This approach also provides upper and lower bounds on the quantiles of $Z_n$. In risk management, these quantiles correspond to the Value at Risk at different probability levels. Such bounds cannot be obtained with the aid of the convex approximation $\tilde{Z}_n$. Indeed, we see from (1.2) that the stop-loss premium of $\tilde{Z}_n$ is an upper bound of the stop-loss premium of $Z_n$; more generally, $E\phi(\tilde{Z}_n)$ is an
upper bound for $E\phi(Z_n)$ for any convex function $\phi$. However, there is in general no relation between $P[Z_n \leq z]$ and $P[\tilde{Z}_n \leq z]$ (since indicator functions are not convex).

Another purpose of this work is to provide several numerical illustrations which enhance the practical interest of our approach. In these illustrations, we will examine the position of the cumulative distribution function corresponding to the convex approximation $\tilde{Z}_n$ in the admissible region delimited by the stochastic bounds on $Z_n$. As a byproduct of our results, the error in the approximation of $Z_n$ by $\tilde{Z}_n$ can be evaluated (in other words, we get an upper bound for the Kolmogorov distance between $Z_n$ and $\tilde{Z}_n$).

2 Stochastic bounds on $Z_n$

In this section, we recall how to build two functions $F_{\text{min}}$ and $F_{\text{max}}$ such that the inequalities

$$F_{\text{min}}(t) \leq P[Z_n \leq t] \leq F_{\text{max}}(t) \text{ for all } t \geq 0,$$

(2.1)

hold, as well as

$$F_{\text{min}}(t) \leq P[\tilde{Z}_n \leq t] \leq F_{\text{max}}(t) \text{ for all } t \geq 0.$$

(2.2)

To this end, we use the following result due to Denuit et al. (1999, Proposition 2). Let $F_1, F_2, \ldots, F_n$ be the respective cumulative distribution functions of $V_1, V_2, \ldots, V_n$. Then, the cumulative distribution function $F_{Z_n}$ of $Z_n = V_1 + V_2 + \ldots + V_n$ is constrained by (2.1) with

$$F_{\text{min}}(t) = \sup_{(v_1, v_2, \ldots, v_n) \in \Sigma(t)} \max \left\{ \sum_{i=1}^{n} P[V_i < v_i] - (n - 1), 0 \right\},$$

and

$$F_{\text{max}}(t) = \inf_{(v_1, v_2, \ldots, v_n) \in \Sigma(t)} \min \left\{ \sum_{i=1}^{n} F_i(v_i), 1 \right\},$$

where

$$\Sigma(t) = \{(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n | v_1 + v_2 + \ldots + v_n = t\}, \ t \in \mathbb{R}.$$

Note that $F_{\text{max}}$ is a bona fide cumulative distribution function, whereas $F_{\text{min}}$ is the left-continuous version of some cumulative distribution function. The bounds in (2.1) and (2.2) are the best-possible bounds on $Z_n$ and $\tilde{Z}_n$ in the sense of stochastic dominance when we know the distribution functions $F_1, F_2, \ldots, F_n$, but no assumption is made on the dependence structure between the $V_i$’s. Equivalently, these bounds hold for all sums (1.1) with given cumulative distribution functions for $V_1, V_2, \ldots, V_n$.

Closed form expressions for the bounds (2.1) can in general not be obtained for distributions of the $V_i$’s and one must resort to numerical evaluation. For more details, see Denuit et al. (1999).
Now, assume we have at our disposal some partial knowledge of the dependence existing between the $V_i$'s, namely that there exists a multivariate cumulative distribution function $G$ satisfying

$$G(v_1, v_2, \cdots, v_n) \leq P[V_1 \leq v_1, V_2 \leq v_2, \cdots, V_n \leq v_n] \text{ for all } v_1, v_2, \ldots, v_n \in \mathbb{R},$$

(2.3)

and a joint decumulative distribution function $\overline{H}$ such that

$$P[V_1 > v_1, V_2 > v_2, \cdots, V_n > v_n] \geq \overline{H}(v_1, v_2, \cdots, v_n) \text{ for all } v_1, v_2, \cdots, v_n \in \mathbb{R}.$$

(2.4)

From Denuit et al. (1999, Proposition 5), the inequalities

$$\sup_{(x_1, x_2, \cdots, x_n) \in \Sigma(t)} G(x_1, x_2, \cdots, x_n) \leq F_{Z_n}(t) \leq 1 - \sup_{(x_1, x_2, \cdots, x_n) \in \Sigma(t)} \overline{H}(x_1, x_2, \cdots, x_n),$$

(2.5)

hold for all $t \in \mathbb{R}$. The bounds in (2.5) are obviously more accurate than those in (2.1).

In the literature, several notions of positive dependence have been introduced in order to express the fact that large values of one of the components of a random vector tend to be associated with large values of the others. In our context, one intuitively feels that in most situations the $V_i$'s mainly “move together” (i.e. a large value of $V_i$ is usually followed by a large value of $V_{i+1}$). For the numerical illustrations in this paper, we will assume that (2.3) and (2.4) are satisfied with

$$G(v_1, v_2, \cdots, v_n) = \prod_{i=1}^{n} F_i(v_i)$$

and

$$\overline{H}(v_1, v_2, \cdots, v_n) = \prod_{i=1}^{n} (1 - F_i(v_i)).$$

In such a case, the $V_i$'s are said to be Positively Orthant Dependent (POD, in short). POD comes thus down to assume that the probability that all the $V_i$’s assume “small” values (i.e. $V_i \leq v_i$, $i = 1, 2, \ldots, n$) is larger than the corresponding probability under the assumption that the $V_i$’s are mutually independent. The interpretation for $\overline{H}$ is similar by substituting “large” for “small”. For more details, see, e.g., Szekli (1995, pp. 144-145).

3 Applications

3.1 Stochastic annuities

Let $\delta_s$ be the force of interest at time $s$ and let $Y_t$ denote the force of interest accumulation function at time $t$, i.e.

$$Y_t = \int_{s=0}^{t} \delta_s ds.$$
The random present value at time 0 of a payment of 1 monetary unit at time \( t \) is given by \( \exp(-Y_t) \), \( t \geq 0 \).

As noticed by Parker (1994b), there are mainly two possible approaches to model the interest randomness, namely the modeling of \( Y_t \) and the modeling of \( \delta_s \). In the first approach, we could let \( Y_t \) be the sum of a deterministic drift of slope \( \delta \) and a perturbation modeled by a Wiener process, i.e.

\[
Y_t = \delta t + \sigma W_t, \quad t \in \mathbb{R}^+,
\]

where \( \sigma \) is a non-negative constant and \( \{W_t, \ t \in \mathbb{R}^+\} \) is a standardized Brownian motion. In such a case, \( V_t \) is log-normally distributed with parameters \(-\delta t \) and \( \sigma^2 t \).

This corresponds to the approach adopted by Goovaerts et al. (1999) who considered a discounted cash flow \( Z_n \) of the form

\[
Z_n = \sum_{i=1}^{n} \exp(-\delta i - X_i),
\]

where the \( X_i \)'s are assumed to be normally distributed with mean 0 and variance \( i\sigma^2 \), and \( \delta \) is the expected force of interest. The convex upper bound \( \tilde{Z}_n \) on \( Z_n \) obtained by Goovaerts et al. (1999) is

\[
\tilde{Z}_n = \sum_{i=1}^{n} \exp \left\{-\delta i - \sigma \sqrt{i} \Phi^{-1}(U)\right\},
\]

where \( \Phi \) is the cumulative distribution function of a standard normal distribution and \( U \) is a random variable uniformly distributed on the unit interval \([0,1]\). The survival function of \( \tilde{Z}_n \) then follows from

\[
P[\tilde{Z}_n > x] = 1 - F_{\tilde{Z}_n}(x) = \Phi(\nu_x),
\]

with \( \nu_x \) the root of the equation

\[
\sum_{i=1}^{n} \alpha_i \exp(-\delta i - \sqrt{i} \sigma \nu_x) = x.
\]

Let us now investigate the accuracy of the bounds (2.1) and (2.5) on the distribution function of \( Z_n \) in the model (3.1). In Figure 1, one sees the functions \( F_{\min} \) and \( F_{\max} \) involved in (2.1), with in between the approximation \( F_{\tilde{Z}_n} \) of the unknown \( F_{Z_n} \) for \( n = 10, \delta = 0.08 \) and \( \sigma = 0.02 \). Figure 3 is the analog for \( n = 20 \). Comparing the cumulative distribution function of the convex approximation (3.2) with the stochastic bounds (2.1), we see from Figures 1 and 3 that (3.2) lies in the very middle of the admissible region bordered by \( F_{\min} \) and \( F_{\max} \). This indicates that (3.2) could be reasonable. In Figures 2 and 4, we further assume that the \( V_i \)'s are POD and we computed the improved bounds furnished in (2.1). Only the lower bound got improved. As it is observed in Example 3 of Denuit et al. (1999), both upper and lower bounds on the distribution of a sum of random variables got improved when the
supports of the random variables are of the form \([a_i, b_i]\) with \(-\infty < a_i < b_i < +\infty\). If \(b_i\) is equal to \(+\infty\) as in Example 1 of Denuit et al. (1999), only the lower bound will be improved with the assumption of POD. In our examples, the random variables are lognormally distributed with supports corresponding to \([0, +\infty)\). If, as in Goovaerts and Dhaene (1999), \(\delta_t\) is defined by a CIR model, then \(Y_t\) will be strictly positive, \(V_t = \exp(-Y_t)\) will take values between 0 and 1, and therefore upper and lower bounds on the distribution of \(Z_t\) will have been improved.

A second approach to model interest randomness is to model \(\delta_s\). For instance, the force of interest can be defined by the differential equation

\[
d\delta_t = -\alpha(\delta_t - \delta)dt + \sigma dW_t,
\]

with non-negative constants \(\alpha\) and \(\sigma\), and with initial value \(\delta_0 = \delta \geq 0\). \(\{\delta_t, \ t \geq 0\}\) is thus an Ornstein-Uhlenbeck process. The force of interest accumulation function \(\{Y_t, \ t \geq 0\}\) is therefore a Gaussian process with mean function

\[
t \mapsto \mu_t = \delta t + (\delta_0 - \delta) \frac{1 - \exp(-\alpha t)}{\alpha},
\]

and autocovariance \((s, t) \mapsto \text{Cov}[Y_s, Y_t] \equiv \omega(s, t)\), where

\[
\omega(s, t) = \frac{\sigma^2}{\alpha^2} \min(s, t) + \frac{\sigma^2}{2\alpha^3} \left\{ -2 + 2\exp(-\alpha s) + 2\exp(-\alpha t) - \exp(-\alpha(t-s)) - \exp(-\alpha(t+s)) \right\}.
\]
Figure 2: Graph of the bounds (2.5) and cumulative distribution function of $\tilde{Z}_{10}$ for (3.1) with $\delta = 0.08$ and $\sigma = 0.02$.

Figure 3: Graphs of the bounds (2.1) and cumulative distribution function of $\tilde{Z}_{20}$ for (3.1) with $\delta = 0.08$ and $\sigma = 0.02$. 
Figure 4: Graphs of the bounds (2.5) and cumulative distribution function of $\tilde{Z}_{20}$ for (3.1) with $\delta = 0.08$ and $\sigma = 0.02$.

see e.g. Parker (1994a, Section 6). Then,

$$Z_n = \sum_{i=1}^{n} \exp(-Y_i),$$

where $Y_i$ is a Normal random variable with mean $\mu_i$ and variance $\omega(i,i)$. In such a case, the convex upper bound $\tilde{Z}_n$ follows from Goovaerts et al. (1999):

$$\tilde{Z}_n = \sum_{i=1}^{n} \exp \left\{ -\mu_i - \sqrt{\omega(i,i)} \Phi^{-1}(U) \right\},$$

where $U$ is a random variable uniformly distributed on the unit interval $[0, 1]$. In Figure 5, you can see the bounds on the cumulative distribution function of $\tilde{Z}_{10}$ in the model (3.3) with $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ and $\sigma = 0.01$, together with the cumulative distribution function of $\tilde{Z}_{10}$. Figure 7 is the analog for $n = 20$. The comments inspired from Figures 1 and 3 still apply. In Figures 6 and 8, we assumed that the $V_i$’s were POD. Again, the improvement with POD is moderate.

### 3.2 Life insurance

Consider a temporary life annuity issued to an individual aged $x$ with curtate-future-lifetime $K$ and denote $P[k < K \leq k + 1] = k\lvert q_x$ and $P[K > n] = np_x$. We assume
Figure 5: Graphs of the bounds (2.1) and cumulative distribution function of $\tilde{Z}_{10}$ for (3.3) with $\delta = 0.06, \delta_0 = 0.08, \alpha = 0.3$ and $\sigma = 0.01$.

Figure 6: Graphs of the bounds (2.5) and cumulative distribution function of $\tilde{Z}_{10}$ for (3.3) with $\delta = 0.06, \delta_0 = 0.08, \alpha = 0.3$ and $\sigma = 0.01$. 
Figure 7: Graphs of the bounds (2.1) and cumulative distribution function of $\tilde{Z}_{20}$ for (3.3) with $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ and $\sigma = 0.01$.

Figure 8: Graphs of the bounds (2.5) and cumulative distribution function of $\tilde{Z}_{20}$ for (3.3) with $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ and $\sigma = 0.01$. 
that $K$ is independent of the random discount factors $V_1, V_2, V_3, \ldots$. The net single premium relating to this contract is given by

$$a^o_{x\,|\,n} = E[a^o_{x\,|\,n}]$$

with

$$a^o_{x\,|\,n} = \begin{cases} 0 & \text{if } K = 0, \\ Z_K & \text{if } K = 1, \ldots, n - 1, \\ Z_n & \text{if } K \geq n, \end{cases}$$

where $Z$ is defined as in (1.1). By conditioning on $K$, the net single premium relating to such a contract is

$$a_{x\,|\,n} = \sum_{k=1}^{n-1} E[Z_k]q_x + E[Z_n]n p_x.$$

The cumulative distribution function of $a^o_{x\,|\,n}$ is also obtained by conditioning on $K$:

$$P[a^o_{x\,|\,n} \leq y] = \sum_{k=1}^{n-1} P[Z_k \leq y]q_x + P[Z_n \leq y]n p_x.$$ 

No explicit expression exists for $P[a^o_{x\,|\,n} \leq y]$, but we use the approach developed above allows us to find stochastic dominance bounds on $a^o_{x\,|\,n}$. In Figure 9, we depicted the graph of the bounds on $P[a^o_{x\,|\,n} \leq y]$ for an individual aged 45 in the model (3.1) with $\delta = 0.08$ and $\sigma = 0.02$. Figure 10 is the analog in model (3.3) with $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ and $\sigma = 0.01$. For these numerical illustrations, we used the standard mortality table (Makeham model) given in Bowers et al. (1996). The bounds in Figures 9 and 10 give a good idea of the danger inherent to the stochastic interest rate combined with the stochastic mortality. Let us mention that the convex approximation of Goovaerts et al. (1999) also applies in this situation.

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**References**


Figure 9: Bounds on $P[\bar{a}^\circ_{x;10} \leq y]$ for $x = 45$ and (3.1) with $\delta = 0.08$ and $\sigma = 0.02$.

Figure 10: Bounds on $P[\bar{a}^\circ_{x;10} \leq y]$ for $x = 45$ and (3.3) with $\delta = 0.06$, $\delta_0 = 0.08$, $\alpha = 0.3$ and $\sigma = 0.01$. 

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