Exact expressions and upper bound for ruin probabilities in the compound Markov binomial model

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Abstract

The compound Markov binomial model was first proposed by Cossette et al. [Scandinavian Actuarial Journal (2003) 301] to introduce time-dependence in the aggregate claim amount increments. As pointed out in [Scandinavian Actuarial Journal (2003) 301], this model can be seen as an extension to Gerber’s compound binomial model. In this paper, we pursue the analysis of the compound Markov binomial model by first showing that the conditional infinite-time ruin probability is a compound geometric tail. Based on this property, an upper bound and asymptotic expression for ruin probabilities are then provided. Finally, special cases of claim amount distributions are considered which allow the exact calculation of ruin probabilities.

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1. Introduction

In this paper, our aim is to further study the conditional and unconditional infinite-time ruin (non-ruin) probabilities in the framework of the compound Markov binomial model. The compound Markov binomial model, an extension to Gerber’s compound binomial model, was first proposed by Cossette et al. (2003) as a discrete-time model which introduces time-dependence in the claim occurrence process. For convenience, a short presentation of the compound Markov binomial model follows. See Cossette et al. (2003) for more details.

In the compound Markov binomial model, time is measured in discrete units 0, 1, … We define the surplus process of an insurance company \{U_k, k \in \mathbb{N}\} by \(U_0 = u\) and

\[ U_k = u + \sum_{j=1}^{k} (c - Y_j), \]

where \(c\) is the premium rate and \(Y_j\) are independent and identically distributed (i.i.d.) claim amounts with common distribution \(F\). The ruin occurs when the surplus process goes negative, i.e., \(U_k < 0\) for some \(k\).

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for \( k \in \mathbb{N}^+ \) where \( u \) corresponds to the initial surplus, \( c \) is the premium rate per period and \( Y_j \) is the eventual claim amount in period \( j \) \((j \in \mathbb{N}^+)\). The premium rate \( c \) is assumed to be equal to 1 and the surplus \( u \) takes values in \( \mathbb{N} \).

We suppose that at most one claim can occur per period. The r.v. \( Y_j \) is then defined as

\[
Y_j = \begin{cases} \mathbb{B}_j, & I_j = 1, \\ 0, & I_j = 0, \end{cases}
\]

where the occurrence r.v. \( I_j \) and the individual claim amount r.v. \( B_j \) are independent in each time period. The r.v. \( I_j \) has a Bernoulli distribution with mean \( q \) and \( B_j \) has a strictly positive discrete r.v. We assume \( \{B_j, j \in \mathbb{N}^+\} \) is a sequence of i.i.d. r.v.’s with probability mass function (p.m.f.) \( f_B \), cumulative distribution function (c.d.f.) \( F_B \), probability generating function (p.g.f.) \( f_B \) and mean \( \mu_B \). Moreover, we assume that the r.v.’s \( I_j \) and \( B_j \) are defined such that

\[ E[Y_j] = qu < c = 1, \quad (1) \]

which ensures that the surplus process \( U_t \) goes almost surely to infinity as \( t \to \infty \), or equivalently, that the infinite-time ruin probabilities go to 0 as \( u \to \infty \).

Let \( \hat{S} = \{S_k, k \in \mathbb{N}^+\} \) be the total claim process where \( S_0 = Y_1 + \cdots + Y_t \). We also define the process \( W = \{W_k, k \in \mathbb{N}^+\} \) by \( W_k = S_k - k \).

When occurrence r.v.’s \( I_j \) are assumed independent, the above model is called the compound binomial model, first proposed by Gerber (1988a,b) and further examined by Shiu (1989), Michel (1989), Willmot (1993), Dickson et al. (1994), Dickson et al. (1995) and De Vylder and Marceau (1996).

In the compound Markov binomial model, we rather suppose that \( \{I_k, k \in \mathbb{N}\} \) is a Markovian process with a two-state transition probability matrix

\[
P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix},
\]

where \( \text{Pr}(I_{k+1} = j|I_k = i) = p_{ij} \) for \( i, j \in \{0, 1\} \) and \( k \in \mathbb{N} \). The transition probability matrix is defined in terms of the probability \( q \) and a correlation parameter \( \pi \)

\[
P = \begin{pmatrix} (1 - q) + \pi q & q - \pi q \\ (1 - q) - \pi(1 - q) & q + \pi(1 - q) \end{pmatrix}
\]

(2)

with initial probabilities \( \text{Pr}(I_0 = 1) = q = 1 - \text{Pr}(I_0 = 0) \), where \( 0 \leq \pi < 1 \) and \( q \in (0, 1) \). Note that \( \{I_k, k \in \mathbb{N}\} \) is sometimes called a Markov–Bernoulli sequence (see Wang, 1981). We can show that the stationary probabilities associated to \( P \) are \( \text{Pr}(I_k = 1) = q \) and \( \text{Pr}(I_k = 0) = 1 - q \) for \( k \in \mathbb{N}^+ \). When \( \pi = 0 \), (2) becomes

\[
P = \begin{pmatrix} 1 - q & q \\ 1 - q & q \end{pmatrix}
\]

which is the transition probability matrix for independent occurrence r.v.’s \( I_k \). Consequently, Gerber’s compound Markov binomial model can be seen as a particular case of the compound Markov binomial model.

Given (2), the introduction of the dependence parameter \( \pi \) leads to a higher probability of a claim free period following a period with no claim (than if a claim has occurred). If, however, a period results in a claim, the probability of observing one in the next period increases. Such an extension can be useful in a context of home insurance issued to a group of homeowners in a given area covering losses resulting from earthquakes. It is well-known that, following a large magnitude earthquake, smaller magnitude earthquakes are expected to be registered near the epicenter of the first one in days following the first event. Additional claim amounts are therefore expected to occur.

In Cossette et al. (2003), we have presented the properties of the Markov–Bernoulli and Markov binomial models and examined the computation of the finite-time and infinite-time ruin probabilities in the framework of this extension. We have also shown the impact of the dependence on the ruin probability and derived a Lundberg exponential bound for such a probability. In the present paper, we focus exclusively on infinite-time ruin probabilities.
In Section 2, we present exact expressions for infinite-time ruin probabilities considering special cases of claim amount distributions. In Section 3, we first show that the conditional infinite-time ruin probabilities in the compound Markov binomial model have an exact compound geometric tail representation. Based on Willmot and Lin (2001), this property of the ruin probabilities allows us to derive both upper bound and asymptotic result for infinite-time ruin probabilities. Finally, an alternative algorithm is proposed to compute ruin probabilities based on Panjer’s algorithm.

2. Ruin probabilities

We are interested in the computation of the ruin probabilities for a given initial surplus \( u \in \mathbb{N} \) which is given by

\[
\psi(u) = \Pr \left( \sup_{k \in \mathbb{N}^+} W_k > u \right) \tag{3}
\]

and its complement, the non-ruin probability, \( \phi(u) = 1 - \psi(u) \). Let us also define the conditional ruin probability \( \psi(u|i) \) as

\[
\psi(u|i) = \Pr \left( \sup_{k \in \mathbb{N}^+} W_k > u|I_0 = i \right).
\]

and its complement, the conditional non-ruin probability \( \phi(u|i) \) as \( \phi(u|i) = 1 - \psi(u|i) \). Clearly, we know that

\[
\psi(u) = (1 - q)\psi(u|0) + q\psi(u|1) \tag{4}
\]

and

\[
\phi(u) = (1 - q)\phi(u|0) + q\phi(u|1). \tag{5}
\]

We assume, throughout this paper, that ruin probabilities hold for infinite-time ruin probabilities.

2.1. Recursive relations

In Cossette et al. (2003), we found recursive relations to compute the conditional non-ruin probabilities in the compound Markov binomial model. In the following proposition, we recall those results in terms of ruin probabilities. We use these results later on to find explicit expressions for ruin probabilities for some special cases of claim amount distributions.

Proposition 1. In the compound Markov binomial model, the conditional ruin probabilities are given by

- for \( u = 0 \):
  \[
  \psi(0|0) = \frac{q}{1 - q} (E[B] - 1),
  \]
  \[
  \psi(0|1) = \frac{\pi(1 - f_B(1)) + \rho_0\psi(0|0)}{\rho_0 - \pi f_B(1)}
  \]

- for \( u \in \mathbb{N}^+ \),
  \[
  \psi(u|0) = \psi(0|0) - \frac{q}{1 - q} \sum_{k=1}^{u} (1 - f_B(k))(1 - \psi(u - k|1)),
  \]

In Cossette et al. (2003), we found recursive relations to compute the conditional non-ruin probabilities in the compound Markov binomial model. In the following proposition, we recall those results in terms of ruin probabilities. We use these results later on to find explicit expressions for ruin probabilities for some special cases of claim amount distributions.
\[
\psi(u|1) = \psi(0|1) - \sum_{k=1}^{u} \frac{p_0(1 - F_B(k)) + \pi f_B(k + 1)}{p_0 - \pi f_B(1)}(1 - \psi(u - k|1)).
\]

(9)

**Proof.** The result is an immediate consequence of Proposition 4 in Cossette et al. (2003) and the fact that
\[
\psi(u|i) = 1 - \phi(u|i).
\]

Simple modifications to (8) and (9), using (6) and (7), lead to
\[
\psi(u|0) = \frac{u}{1 - q} \sum_{k=1}^{u} (1 - F_B(k)) \psi(u - k|1) + \frac{u}{1 - q} \sum_{k=u+1}^{\infty} (1 - F_B(k))
\]
and
\[
\psi(u|1) = \sum_{k=1}^{u} \frac{p_0(1 - F_B(k)) + \pi f_B(k + 1)}{p_0 - \pi f_B(1)} \psi(u - k|1) + \sum_{k=1}^{u} \frac{p_0(\mu_B - 1) + \pi(1 - f_B(1))}{p_0 - \pi f_B(1)}
\]
\[
\sum_{k=u+1}^{\infty} \frac{p_0(1 - F_B(k)) + \pi f_B(k + 1)}{p_0 - \pi f_B(1)} \psi(u - k|1) + \sum_{k=1}^{u} \frac{p_0(1 - F_B(k)) + \pi f_B(k + 1)}{p_0 - \pi f_B(1)}
\]

(10)

and

(11)

for \(u \in \mathbb{N}^+\). In the next sub-sections, we use (10) and (11) to find explicit expressions for ruin probabilities considering specific claim amount distributions.

2.2. Special cases of claim amount distributions

In risk theory, it is rare to find exact expressions for ruin probabilities. It is therefore interesting to identify distributions of the individual claim amount leading to such expressions. It has been previously shown that, in Gerber’s compound binomial model (e.g. Willmot, 1993), an explicit expression for the ruin probabilities can be found when \(B \in \{2\}, B \in \{1, 2\}\) and \(B\) follows a zero-truncated geometric distribution. In this section, we show that these distributions also admit explicit expressions for the ruin probabilities in the framework of the compound Markov binomial model.

2.2.1. First case: \(B \in \{2\}\)

We first assume that \(B\) can only take the value 2. In the present case, the process \(W\) can be seen as a special random walk since \(W\) can either decrease or increase by one unit from its previous level in each time period. Here however, the increments are no longer independent. Explicit expressions for the ruin probabilities in this context are given in the following proposition.

**Proposition 2.** For the compound Markov binomial model with \(B \in \{2\}\), the conditional and unconditional ruin probabilities are given by
\[
\psi(u|i) = \psi(0|i) \left( \frac{p_{11}}{p_{01}} \right)^u, \quad i \in \{0, 1\},
\]
and
\[
\psi(u) = \psi(0) \left( \frac{p_{11}}{p_{00}} \right)^u,
\]
for \(u \in \mathbb{N}^+\).
Proof. The proof is omitted since the present case is a special case of $B \in \{1, 2\}$ which is studied in the Section 2.2.2.

Remark 1. If $\pi = 0$, we obtain the expression for the ruin probability defined in the context of the gambler’s problem $\psi(u) = \psi(0)(1 - q)^u$ for $u \in \mathbb{N}^+$ (see e.g. Ross, 2000; Rolski et al., 1999; Willmot, 1993).

2.2.2. Second case: $B \in \{1, 2\}$

Assume now that the r.v. $B$ takes values in $\{1, 2\}$. It follows that increments $(Y_k - 1)$ of the process $W$ can either take the value of $-1, 0$ or $1$. In the following proposition, we provide the explicit expressions for the ruin probabilities.

Proposition 3. In the compound Markov binomial model with $B \in \{1, 2\}$, the conditional and unconditional ruin probabilities are given by

$$\psi(u | i) = \psi(0 | i) \left( \frac{p_{11}f_B(2)}{p_{00} - \pi f_B(1)} \right)^u, \quad i \in \{0, 1\},$$

(12)

and

$$\psi(u) = \psi(0) \left( \frac{p_{11}f_B(2)}{p_{00} - \pi f_B(1)} \right)^u,$$

(13)

for $u \in \mathbb{N}^+$.

Proof. Since $B \in \{1, 2\}$, (10) and (11) become, respectively

$$\psi(u | 0) = \frac{q}{1 - q}(1 - f_B(1))\psi(u - 1 | 1)$$

(14)

and

$$\psi(u | 1) = \frac{p_{01}(1 - f_B(1)) + \pi f_B(2)}{p_{00} - \pi f_B(1)} \psi(u - 1 | 1) = \frac{p_{11}f_B(2)}{p_{00} - \pi f_B(1)} \psi(u - 1 | 1),$$

(15)

for $u \in \mathbb{N}^+$. From (15), one deduces

$$\psi(u | 1) = \left( \frac{p_{11}f_B(2)}{p_{00} - \pi f_B(1)} \right)^u \psi(0 | 1),$$

for $u \in \mathbb{N}^+$, which implies that (14) can be rewritten as

$$\psi(u | 0) = \frac{q}{1 - q}(1 - f_B(1)) \left( \frac{p_{11}f_B(2)}{p_{00} - \pi f_B(1)} \right)^{u-1} \psi(0 | 1).$$

(16)

Moreover, since $B \in \{1, 2\}$, (6) and (7) can be simplified to

$$\psi(0 | 0) = \frac{q}{1 - q} f_B(2)$$

and

$$\psi(0 | 1) = \frac{p_{11}f_B(2)}{p_{00} - \pi f_B(1)},$$

which implies that (16) becomes

$$\psi(u | 0) = \left( \frac{p_{11}f_B(2)}{p_{00} - \pi f_B(1)} \right)^u \psi(0 | 0).$$

(13) is a direct consequence of (12) at both $i = 0$ and $i = 1$ and (4).
Remark 2. If \( \pi = 0 \), (13) becomes \( \psi(u) = \psi(0)(qf_B(2)/(1-q))^u \) for \( u \in \mathbb{N}^+ \), which is the explicit expression for the ruin probability defined in the framework of the compound binomial model (see Willmot, 1993).

2.2.3. Third case: zero-truncated geometric distribution

Finally, we consider the case where \( B \) has a zero-truncated geometric distribution with p.m.f. \( f_B(i) = (1-\alpha)\alpha^{i-1} \) and c.d.f. \( F_B(i) = 1-\alpha^i \) for \( i \in \mathbb{N}^+ \). The mean of \( B \) is \((1-\alpha)^{-1}\). The following proposition provides the explicit expressions for the ruin probabilities in this case.

Proposition 4. In the compound Markov binomial model where \( B \) has a zero-truncated geometric distribution, the conditional and unconditional ruin probabilities are given by

\[
\psi(u | i) = \psi(0 | i) \left( \frac{\alpha}{p_{00} - \pi(1-\alpha)} \right)^u, \quad i \in \{0, 1\},
\]

and

\[
\psi(u) = \psi(0) \left( \frac{\alpha}{p_{00} - \pi(1-\alpha)} \right)^u,
\]

for \( u \in \mathbb{N}^+ \).

Proof. For the zero-truncated geometric case, (11) can be simplified to

\[
\psi(u | 1) = \sum_{k=1}^{u} \frac{p_{01} + \pi(1-\alpha)}{p_{00} - \pi(1-\alpha)^k} \psi(u - k | 1) p_{00} - \pi f_B(1) + \sum_{k=1}^{\infty} \frac{p_{01} + \pi(1-\alpha)}{p_{00} - \pi(1-\alpha)^k} \psi(u - k | 1) p_{00} - \pi f_B(1).
\]

and, similarly,

\[
\psi(u + 1 | 1) = \sum_{k=1}^{u+1} \frac{p_{01} + \pi(1-\alpha)}{p_{00} - \pi(1-\alpha)^k} \psi(u + 1 - k | 1) p_{00} - \pi f_B(1) + \sum_{k=1}^{\infty} \frac{p_{01} + \pi(1-\alpha)}{p_{00} - \pi(1-\alpha)^k} \psi(u + 1 - k | 1) p_{00} - \pi f_B(1).
\]

Multiplying (19) by \( \alpha \) and then, subtracting it from (20), yields

\[
\psi(u + 1 | 1) - \alpha \psi(u | 1) = \frac{\alpha(p_{01} + \pi(1-\alpha))}{p_{00} - \pi f_B(1)} \psi(u | 1)
\]

which can be rewritten as

\[
\psi(u + 1 | 1) = \frac{\alpha}{p_{00} - \pi f_B(1)} \psi(u | 1).
\]

We deduce from (21) that

\[
\psi(u | 1) = \psi(0 | 1) \left( \frac{\alpha}{p_{00} - \pi f_B(1)} \right)^u,
\]

for \( u \in \mathbb{N} \) which completes the proof of (17) for \( i = 1 \).

Moreover, for the zero-truncated geometric distribution, it follows from (10) and (11) that

\[
\psi(u | 1) = \left( \frac{1-\alpha}{q} \right) \left( \frac{p_{01} + \pi(1-\alpha)}{p_{00} - \pi f_B(1)} \right) \psi(u | 0).
\]

Combining (22) with (23) yields

\[
\psi(u) = \left( \frac{q}{1-q} \right) \left( \frac{p_{01} + \pi(1-\alpha)}{p_{00} - \pi f_B(1)} \right)^u \psi(0) = \left( \frac{\alpha}{p_{00} - \pi f_B(1)} \right)^u \psi(0),
\]

which corresponds to (17) for \( i = 0 \). (18) is a direct consequence of (17) at both \( i = 0 \) and \( i = 1 \) and (4).
Remark 3. Letting $\pi = 0$ in (18), we obtain $\psi(u) = \psi(0)(u/(1 - q))^a$ for $u \in \mathbb{N}^+$ which corresponds to the explicit expression for the ruin probability defined in the compound binomial model (see Willmot, 1993).

3. Upper bound and asymptotic expression for ruin probabilities

3.1. Introduction

Cossette et al. (2003) propose the following upper bound to the ruin probabilities in the compound Markov binomial model

$$\psi(u) \leq e^{-R^*u}, \quad u \in \mathbb{N},$$

(24)

which is a Lundberg-type upper bound similar to those derived, first, in the classical risk model and later, in a discrete framework, the compound binomial model. The adjustment coefficient $R^*$ is defined as

$$R^* = \min(R_0, R_1)$$

where, for each $i \in \{0, 1\}$, $R_i$ is the strictly positive solution to $p_{i0} + p_{i1}E[e^{RB}] = e^{Ri}$ which exists if $p_{11} \mu_B < 1$.

It is shown in Cossette et al. (2003) that $R^* = R_1$. The problem with the Lundberg-type upper bound (24) is that $R_1$ (i.e. $R^*$) exists only for values of $\pi$ in $[0, (1 - q\mu_B)/(1 - q)]$. Therefore, for cases where the relative security margin $\eta > 0$ is low, the permissible interval for $\pi$ is narrower. Furthermore, for a low relative security margin and a high dependence parameter $\pi$, the upper bound does not exist.

To get around this problem, we determine, in this section, a more general upper bound which is valid for any combination of positive relative security margin and dependence parameter $\pi \in [0, 1)$. We begin by showing that the conditional ruin probabilities in the compound Markov binomial model have an exact compound geometric tail representation as the unconditional ruin probabilities in the compound binomial model (see Willmot, 1993). Based on this property and given some constraints on claim amount distributions, we show that the ruin probabilities can be dominated by the survival function of a geometric distribution and asymptotically behave as the tail of a geometric distribution.

3.2. Exact compound geometric tail representation

In this section, we show that $\{\psi(u), u \in \mathbb{N}\}$ for $i \in \{0, 1\}$ is a compound geometric tail i.e. that $\psi(u)$ can be expressed as the survival function of a given compound geometric distribution. For further use, let us define the r.v. $W$ with a compound geometric distribution as

$$W = \sum_{M=0}^{\infty} X_M, \quad M > 0,$$

$$M = 0,$$

where $M$ has a geometric distribution, defined on $\mathbb{N}$, with p.m.f. $p_M(x) = \gamma x(1 - \gamma)$ and p.g.f. $\tilde{p}_M(z) = (1 - \gamma)(1 - \gamma z)$. Also, $X_1, X_2, \ldots$ is a sequence of discrete and strictly positive i.i.d. r.v.'s, independent of $M$, and with p.m.f. $p_X c.d.l. P_X$ and p.g.f. $\tilde{p}_X$. It follows that

$$E[W] = E[M]E[X_1] = \frac{\gamma}{1 - \gamma}E[X_1],$$

(25)

$$P(W > x) = \sum_{r=1}^{\infty} \gamma^r(1 - \gamma)\tilde{p}_X^r(x)$$
and
\[ \hat{p}_W(z) = \hat{p}_M(\hat{p}_X(z)) = \frac{1 - \gamma}{1 - \gamma \hat{p}_X(z)} \]
where \( P^k_\gamma(x) \) is the survival function associated to the \( k \)-fold convolution of the \( P_\gamma \).

For the remaining of this section, we assume that the distribution of \( \hat{d} \) admits a p.g.f. \( \hat{f}_B(z) \) for \( z \geq 1 \). We begin with a definition based on Willmot (1993).

**Definition 1.** We define, respectively, \( \phi(z, t_1), \hat{\phi}(z, t_2) \) and \( \hat{\phi}(0, t_1) \) as \( \phi(z, t_1) = \sum_{i=0}^{\infty} z^i \phi(u, t_1), \hat{\phi}(z, t_2) = \sum_{k=0}^{\infty} z^k \hat{\phi}(k|i) \) and \( \hat{\phi}(0, t_1) = \sum_{i=0}^{\infty} z^i \phi(0, k|i) \).

For sake of completeness, we recall the definition of an exact compound geometric tail representation.

**Definition 2.** We say that the sequence \( \{a(n), n \in \mathbb{N} \} \) with generating function \( \hat{a} \) is a compound geometric tail if \( \hat{a} \) can be written as \( \hat{a}(z) = (1 - \hat{p}_W(z))/(1 - z) \). In this case, we have
\[ a(n) = P(W > n), \quad n \in \mathbb{N} . \]

**Proposition 5.** In the compound Markov binomial model, the conditional ruin probabilities given by
\[ \phi(u|i) = \sum_{k=0}^{\infty} (1 - \phi(0))((\phi(0)i)kG^k_{\gamma}(u), \] where \( G^k_{\gamma} \) and \( G^k_{\gamma} \) are, respectively, the survival function associated to the \( k \)-fold convolution of the p.g.f. \( G_{\gamma}(1 - z) = (p_{11} - 1 - z)p_{10} + (1 - 1 - z)p_{01} + (1 - 1 - z)p_{00} \) and \( G_{\gamma}(1 - z)p_{11} = (1 - z)p_{01} + (1 - 1 - z)p_{00} \).

The generating function of the ruin probabilities \( \hat{\phi}(z) \) can be written as
\[ \hat{\phi}(z) = \frac{1 - R(z)}{1 - z} , \] where \( R(z) \) for \( i = 0 \) and \( i = 1 \) is the p.g.f. of a compound geometric distribution.

**Proof.** First, we denote by \( \hat{\phi}(z|i) \) the generating function of the conditional ruin probabilities \( \{\phi(u|i), u \in \mathbb{N} \} \) for which
\[ \hat{\phi}(z) = \sum_{u=1}^{\infty} z^u \phi(u|i) = \sum_{u=1}^{\infty} z^u (1 - \phi(u|i)) = \frac{1 - (1 - z)\hat{\phi}(z|i)}{1 - z} = \frac{1 - R(z)}{1 - z} , \]
where \( \hat{\phi}(z|i) \) holds for the generating function of the conditional non-ruin probabilities \( \{\phi(u|i), u \in \mathbb{N} \} \) and \( R(z) = (1 - z)\hat{\phi}(z) \).

Our goal is to prove that \( R(z) \) is the p.g.f. of a compound geometric distribution. For that purpose, we first need the expression for \( \hat{\phi}(z|i) \).

By conditioning on the claim occurrence and the claim amount r.v.'s during the first time period, we know that
\[ \phi(u - 1, k|i) = \rho_{ij} \phi(u, k - 1|i) + \rho_{ij} \sum_{j=1}^{w} \phi(u - j, k - 1|1) f_k(j) , \]
for \( i \in \{0, 1\} \). Multiplying (30) by \( z^u \) and summing for \( u = 1, 2, \ldots \) yields
\[ \sum_{u=1}^{\infty} z^u \phi(u - 1, k|i) = \sum_{u=1}^{\infty} z^u \left( \rho_{ij} \phi(u, k - 1|i) + \rho_{ij} \sum_{j=1}^{w} \phi(u - j, k - 1|1) f_k(j) \right) \]
which becomes, combined with Definition 1
\[ z^k \phi(z, k) = p_0 \phi(z, k-1) + \phi(0, k-1) + p_1 \phi(z, k-1) \]  
(31)

Similarly, multiplying (31) by \( r^k \) and summing for \( k = 1, 2, \ldots \) results in
\[ z(\phi(z, r) - \phi(z, 0)) = t p_0 (\phi(z, r) - \phi(0, r)) + t p_1 (\phi(z, r) - \phi(0, r)) \]  
(32)

using Definition 1. Note that \( \phi(z, 0) = 1/(1 - z) \).

Based on a Tauberian theorem for power series (see Feller, 1971, p. 447), one has
\[ \hat{\phi}(z) = -\lim_{r \to 1} (t - 1) \phi(z, r) \]  
(33)

and
\[ \phi(0) = -\lim_{r \to 1} (t - 1) \phi(0, r) \]  
(34)

which are similar to the results obtained in Willmot (1993) in the compound binomial model. Therefore, taken
\[ \lim_{r \to 1} \] on both sides of (32) which have previously been multiplied by \( (t - 1) \) yields
\[ z^k \hat{\phi}(z) = p_0 \hat{\phi}(z, 0) + \phi(0, 0) + \phi(0, 1) \]  
(35)

Using (33) and (34). Combining (35) at both \( i = 0 \) and \( i = 1 \) yields
\[ \hat{\phi}(0) = \frac{\pi \hat{f}(z) - \pi \phi(0)}{(z - \pi \hat{f}(z)) (z \pi \hat{f}(z) - \pi \phi(0))} \]  
(36)

and
\[ \hat{\phi}(1) = \frac{z \pi \phi(0) - (z - \phi(0)) (z \pi \phi(0) - \pi \phi(0))}{(1 - \phi(0)) (z \pi \phi(0) - \pi \phi(0))} \]  
(37)

We recall that our goal was to show that \( R_i(z) = (1 - z^k) \hat{\phi}(z) \) is the p.g.f. of a compound geometric distribution for both \( i = 0 \) and \( i = 1 \). Based on (36) and (37), \( R_0(z) \) and \( R_1(z) \) can, respectively, be written as
\[ R_0(z) = \frac{1 - \phi(0)}{(1 - z)(1 - (z \pi \phi(0) - \pi \phi(0)) \pi \hat{f}(z) - \pi \phi(0))} \]  
(38)
and

\[
R_i(z) = \frac{1 - \psi(0|1)}{1 - \psi(0|1)} \left( \frac{1}{1 - z} \right) \left( 1 - ((p_{11} f_0(z) - z) + \pi f_0(1 - z) f_0(1) - \nu_0) \right) \left( 1 - \psi(0|1) \right) = 1 - \frac{(1/1 - z)(1 - z)f_0(1) - \nu_0 + (p_{11} - \pi/2) f_0(1)}{1 - \psi(0|1)} = 1 - \frac{(1/1 - z)(1 - z)f_0(1) - \nu_0 + (p_{11} - \pi/2) f_0(1) - \nu_0}{1 - \psi(0|1)} \\
\]

(39)

Letting

\[
P_0(z) = \frac{z}{1 - z} \frac{\tilde{f}_0(z) - z}{\pi f_0(1) - \nu_0} \quad \text{and} \quad P_1(z) = \pi f_0(1 - z) \frac{(p_{11} - \pi/2) f_0(1) - \nu_0}{\pi f_0(1) - \nu_0},
\]

(40) and (41) can be written simultaneously as

\[
R_i(z) = \frac{1 - \psi(0|1)}{1 - \psi(0|1)} \left( \frac{1}{1 - z} \right) \left( 1 - ((p_{11} f_0(z) - z) + \pi f_0(1 - z) f_0(1) - \nu_0) \right) \left( 1 - \psi(0|1) \right) = 1 - \frac{(1/1 - z)(1 - z)f_0(1) - \nu_0 + (p_{11} - \pi/2) f_0(1) - \nu_0}{1 - \psi(0|1)} \\
\]

(42)

Simple modifications to (42) lead to

\[
P_k(z) = \frac{\pi f_0(1) - \nu_0 \sum_{k=1}^{\infty} z^k (1 - F_0(k)) + \pi f_0(1) - \nu_0}{\pi f_0(1) - \nu_0} \\
\]

(43)

We denote by \( p_1 \) the mass function related to the \( P_1(z) \) which is defined as

\[
p_1(k) = \begin{cases} 
\frac{\nu_0 (1 - F_0(k)) + \pi f_0(k + 1)}{\pi f_0(1)} & \text{if } k \in \mathbb{N}^+, \\
0 & \text{elsewhere}
\end{cases}
\]

(44)
Let us denote by $p \in \{2001, \text{pp. 124–125}\}$ respectively, where $	ilde{\psi}(u)$, $u \in \mathbb{R}$, which will allow us to simplify the obtention of this upper bound.

### 3.3. Upper bound

The derivation of the upper bound for the conditional and unconditional ruin probabilities is based on the exact compound geometric representation of $\{\tilde{\psi}(u), u \in \mathbb{N}\}$ for $i \in \{0, 1\}$. We begin with a definition and a lemma which will allow us to simplify the obtention of this upper bound.

**Definition 3.** Denote by $z_0$ and $z_1$ the solutions strictly greater than 1 (if they exist) to

\[
p_{00} + p_{01} F_B(z) = z - \pi f_B(z) (1 - \frac{1}{z})
\]

and

\[
p_{10} + p_{11} \tilde{F}_B(z) = z - \pi (1 - \frac{\tilde{f}_B(z)}{z})
\]

respectively, where $\tilde{F}_B(z)$ exists for any $z \geq 1$. 

\[\psi(u), u \in \mathbb{R}\]

where $F_B(z) = \psi(z) + \pi f_B(z)$ exists for any $z \geq 1$ and $\tilde{F}_B(z)$ is the p.g.f. of a compound geometric distribution with $\gamma = \pi f_B(z)$ and $F_B(z)$ is the p.g.f. of a compound geometric tail as $\{\psi(u), u \in \mathbb{N}\}$ in the compound binomial model (see Willmot and Lin, 2001, pp. 124–125).

\[\psi(u), u \in \mathbb{R}\]

\[\psi(u), u \in \mathbb{R}\]

\[\psi(u), u \in \mathbb{R}\]
In the following lemma, we first show that if the solutions \( z_0 \) and \( z_1 \) to (48) and (49) exist then they are identical. In the second part, we prove that, if \( 1 > q \mu_B \), as it is the case in the compound Markov binomial model, \( z_0 \) and \( z_1 \) are unique and both strictly greater than 1.

**Lemma 1.**

(a) If the solutions \( z_0 \) and \( z_1 \) to (48) and (49) exist, then \( z_0 = z_1 = z^* \).

(b) If \( 1 > q \mu_B \), \( z^* \) is unique and strictly greater than 1.

**Proof.**

(a) We first rearrange (48) as

\[
p_{00} + p_{11} \tilde{f}_B(z) = z + \pi \frac{\tilde{f}_B(z)}{z}
\]

(50)

and by subtracting \( \pi \) on both sides of Eq. (50) we obtain (49). Therefore, (48) and (49) induce the same solution.

In addition, it is straightforward that \( m(1) = n(1) \) which implies that functions \( m \) and \( n \) have at least one intersection point. Moreover, given that

\[
\lim_{z \to \infty} \frac{m(z)}{n(z)} = \lim_{z \to \infty} \frac{p_{00} + p_{11} \tilde{f}_B(z)}{z + \pi \frac{\tilde{f}_B(z)}{z}} = \infty
\]

and \( \lim_{z \to 0^+} \frac{m(z)}{n(z)} = \frac{p_{00}}{\pi f_B(1)} > 1 \) since \( \pi f_B(1) \leq \pi \leq 1 - q + \pi q \), we know that there exists only one other intersection point between functions \( m \) and \( n \). For this intersection point to be greater than 1, one must show that

\[
m'(1) < n'(1).
\]

(51)

Since \( m'(1) = p_{11} \mu_B \) and \( n'(1) = 1 + \pi (\mu_B - 1) \), it follows that (51) is satisfied if \( 1 > q \mu_B \).

(b) Using (50), we define the functions \( m \) and \( n \) as

\[
m(z) = p_{00} + p_{11} \tilde{f}_B(z)
\]

and

\[
n(z) = z + \pi \frac{\tilde{f}_B(z)}{z}
\]

for \( z > 0 \). With routine calculations, one can show that \( m \) and \( n \) are two strictly increasing and convex functions on \((0, \infty)\).

In Lemma 1 and Proposition 5, results relative to the compound geometric distribution (see, e.g. Willmot and Lin, 2001) can be applied. In the next proposition, we present upper bounds to the ruin probabilities in the framework of the compound Markov binomial model.

**Proposition 6.** In the compound Markov binomial model, the conditional and unconditional ruin probabilities satisfy

\[
\psi(u|i) \leq \left( \frac{1}{z} \right)^{i+1}, \quad i \in \{0, 1\}
\]

(52)

and

\[
\psi(u) \leq \left( \frac{1}{z} \right)^{u+1}, \quad u \in \mathbb{N}
\]

(53)

for \( u \in \mathbb{N} \).

**Proof.** Since \( \{\psi(u|i), u \in \mathbb{N}\} \) for \( i \in \{0, 1\} \) is a compound geometric tail and from Corollary 7.2.1 of Willmot and Lin (2001), it follows that
\[
\psi(u|\mathcal{I}) \leq \left( \frac{1}{z^*} \right)^{u+1},
\]

where \(z_1 > 1\) is the solution to
\[
P_i(z_1) = \frac{1}{\psi(0)}.
\]

(55) is equivalent to finding, respectively, \(z_0 > 1\) and \(z_1 > 1\) such that
\[
p_{00} + p_{01}\tilde{f}_B(z) = z - \pi\tilde{f}_B(z) \left( 1 - \frac{1}{z} \right)
\]
and
\[
p_{10} + p_{11}\tilde{f}_B(z) = z - \pi \left( 1 - \frac{\tilde{f}_B(z)}{z} \right).
\]

Eqs. (56) and (57) correspond to relations (48) and (49) in the definitions of \(z_0^*\) and \(z_1^*\). In Lemma 1, we show that the solutions of (56) and (57) are the same (i.e. \(z_0^* = z_1^* = z^*\)) and \(z^* > 1\) is unique. We conclude that (54) becomes
\[
\psi(u|\mathcal{I}) \leq \left( \frac{1}{z^*} \right)^{u+1}, \quad i \in \{0, 1\},
\]
from which, combined with (4), we deduce that \(\psi(u) \leq (1/z^*)^{u+1}\).

Refinements to the upper bound (52) and other results can be obtained for conditional ruin probabilities if the p.g.f. \(P_0(z)\) and \(P_1(z)\) have some particular characteristics. We refer the reader to Willmot and Lin (2001) for more details.

3.4. Asymptotic expression

We provide, in the next proposition, asymptotic expressions for the conditional ruin probabilities.

**Proposition 7.** In the compound Markov binomial model, asymptotic expressions for \(\psi(u|0)\) and \(\psi(u|1)\) are, respectively, given by
\[
\psi(u|0) \sim \frac{(1 - \psi(0))(p_{00} - \pi\tilde{f}_B(z^*)z^*)}{(p_{11}\tilde{f}_B(z^*) - 1 - (\pi\tilde{f}_B(z^*)/z^*))} \left( \frac{1}{z^*} \right)^{u+1},
\]
and
\[
\psi(u|1) \sim \frac{(1 - \psi(1))(p_{10} - \pi f_B(1))}{(p_{11}f_B(z^*) - 1 - (\pi f_B(z^*)/z^*))} \left( \frac{1}{z^*} \right)^{u+1}.
\]

**Proof.** In order to derive an asymptotic result for the conditional ruin probabilities, we rewrite (40) and (41), respectively, as
\[
\psi(0|0)(1 - z)(\pi\tilde{f}_B(z) - \zeta\psi(0))P_0(z) = \zeta\psi(0)\tilde{f}_B(z) - z
\]
and
\[
\psi(0|1)(1 - z)(\pi f_B(1) - p_{00})P_1(z) = \pi f_B(1)(1 - z) - \zeta\psi(0) + \left( p_{11} - \frac{\pi}{z} \right)\tilde{f}_B(z).
\]
Differentiation of (60) and (61) with respect to $z$ yields

$$P_0'(z) = \frac{(p_{01}(\tilde{f}_B(z) - z) + zp_{01}(\tilde{f}_B(z) - 1)}/\psi(0|z) + P_0(z)(\pi f_1(z) - zp_0 - (1 - z)(\pi f_0(z) - p_0))}{(1 - z)(\pi f_0(z) - zp_0)} \psi(0|z)$$

and

$$P_1'(z) = \frac{(1/\psi(1|z))(\pi/z^{2})f_0(z) + p_{11} - (\pi/z)f_1'(z)}{(1 - 1/p_0 - \pi f_1(1))}$$

which become, when evaluating at $z = z^*$

$$\psi(0|z^*) = \frac{p_{01}(\tilde{f}_B(z^*) - 1 - (\pi/z^{*})(\tilde{f}_B(z^*) - (\tilde{f}_B(z^*)/z^{*}))}{\rho_0 - \pi f_0(z^*)/z^{*}}$$

and

$$\psi(0|z^*) = \frac{p_{11}(\tilde{f}_B(z^*) - 1 - (\pi/z^{*})(\tilde{f}_B(z^*) - (\tilde{f}_B(z^*)/z^{*}))}{\rho_0 - \pi f_0(1)}.$$  (62)

Applying Theorem 2 of Willmot (1989), one concludes that

$$\psi(u) \sim 1 - \frac{1}{\psi(0|z^*)} + \frac{1}{\psi(0|z^*)} \left( \frac{1}{z^*} \right)^{u+1},$$  (64)

for $i \in \{0, 1\}$ as $u \to \infty$. Combining (62)–(64) at $i = 0$ and (63) and (64) at $i = 1$ yields (58) and (59), respectively.

### 3.5 Numerical illustration

In the following example, exact values of ruin probabilities are compared (in Table 1) to the Lundberg-type upper bound (if it exists) and the more general upper bound presented in this paper. Asymptotic values for the ruin probabilities are provided in Table 2. We assume that $r.v.'s \ B_k$ have a zero-truncated geometric distribution with mean $8$ and p.m.f. $f_B(j) = (1 - 7/8)^{j-1}(7/8)^{j}$ for $j \in \mathbb{N}^+$. We assume that $\Pr(I_k = 1) = q = 0.07$. We consider three cases for $\pi$: $\pi = 0$ (independence), $\pi = 0.05$ and $\pi = 0.2$. In such case, the Lundberg-type upper bound exists only for values of $\pi$ in $[0, 0.0591]$.

<table>
<thead>
<tr>
<th>$u$</th>
<th>$\pi = 0$</th>
<th>$\pi = 0.05$</th>
<th>$\pi = 0.2$</th>
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<td>G.B.</td>
<td>$\psi(u)$</td>
</tr>
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<td>0.3954</td>
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<td>0.0051</td>
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<tr>
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<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
</tbody>
</table>
The results in Tables 1 and 2 allow the following observations and comments:

- With a relative security margin of 78.6% which is quite unrealistic in an insurance context, the Lundberg-type upper bound exists only for values of $\pi$ in $[0, 0.0591)$ which is an enormous constraint on the $\pi$ value in the compound Markov binomial model. Most importantly, this interval shrinks when the relative security margin decreases (to be more realistic).

- The results contained in Table 1 seem to indicate that the more general upper bound gives a tighter upper bound than the Lundberg-type one. This conclusion is always verified by comparing (24) with (53) from which it can be proven that $z^* \geq e^{R^*}$ (when $R^*$ exists). In the case where $\pi = 0$, the solutions $z^*$ and $e^{R^*}$ are identical. However, (53) is still tighter than (24) due to their respective exponents.

- In the continuous Poisson/exponential model, Cramer’s asymptotic formula gives the exact value for the ruin probabilities. This is also true in its discrete equivalent, i.e. the compound binomial model with a zero-truncated geometric distribution for the individual claim amount. The results of Table 2 seem to indicate that this is also valid within the compound Markov binomial model framework. This is formally proven in the proposition that follows.

**Proposition 8.** In the compound Markov binomial model when $B$ a zero-truncated geometric distribution with parameter $\alpha$, the asymptotic expressions (58) and (59) correspond to the exact values for the conditional ruin probabilities found in (17), respectively, at $i = 0$ and $i = 1$.

**Proof.** Since $B$ has a zero-truncated geometric distribution with p.g.f. $\tilde{f}_B(z) = (1 - \alpha)z/1 - \alpha z$, (48) can be simplified to

$$p_{00}(1 - z^*) = \alpha z^*(1 - z^*) + \pi(1 - \alpha)(1 - z^*)$$

which is equivalent to

$$z^* = \frac{p_{00} - \pi(1 - \alpha)}{\alpha}$$  \hspace{1cm} (65)

Note that, in such a case, $z^*$ has an analytical form. From (65), (58) and (59) become, respectively

$$\psi(u|0) \sim \frac{(1 - \psi(0|0))p_{00} - \pi(\tilde{f}_B(z^*)/z^*))}{z^*(p_{11} f_B(z^*) - 1 - \pi(\tilde{f}_B(z^*) - (f_B(z^*)/z^*))} \left(\frac{\alpha}{p_{00} - \pi(1 - u)}\right)^n$$  \hspace{1cm} (66)
Using (71), the last equality becomes
\[ z^i(p_1 f_k(z^i) - 1) = p_i f_k(z^i) - (f_k(z^i)/z^i) \]
which corresponds to (70).

Therefore, by comparing (17) at both \( i = 0 \) and \( i = 1 \), respectively, to (66) and (67), it only remains to be proven that

\[ \psi(0) = \frac{(1 - \psi(0)) (p_{00} - \pi f_{00}(1))}{z^i(p_{11} f_{k}(z^i) - 1) - \pi f_{k}(z^i) - (f_{k}(z^i)/z^i)} \]

and

\[ \psi(1) = \frac{(1 - \psi(1)) (p_{10} - \pi f_{10}(1))}{z^i(p_{11} f_{k}(z^i) - 1) - \pi f_{k}(z^i) - (f_{k}(z^i)/z^i)} \]

to complete the proof. But before proving (68) and (69), one easily deduces from (6) and (7), in the case where \( B \) has a zero-truncated geometric distribution, that

\[ \psi(0) = \frac{q}{1 - a} \frac{p_{11} - \pi a}{p_{00} - \pi (1 - a)} \]

and

\[ \psi(1) = \frac{a}{1 - a} \frac{p_{11} - \pi a}{p_{00} - \pi (1 - a)} \]

We begin by proving (69). Since \( f_{k}(z) = (1 - a)/(1 - az)^2 \) and from (65), (69) can be rewritten as

\[ \psi(0) = \frac{(1 - \psi(0)) (p_{00} - \pi f_{00}(1))}{z^i(p_{11} f_{k}(z^i) - 1) - \pi f_{k}(z^i) - (f_{k}(z^i)/z^i)} \]

which is equivalent to

\[ \psi(0) = \frac{a}{1 - a} \left( \frac{p_{11} - \pi a}{(1 - az)^2} - 1 \right)^{-1} \]

Since \( 1 - az^i = p_{00} + \pi (1 - a) = p_{11} - \pi a \), (72) becomes

\[ \psi(0) = \frac{1 - az^i}{1 - a} \frac{a}{1 - a} \frac{p_{11} - \pi a}{p_{00} - \pi (1 - a)} \]

which corresponds to (71).

To prove (68), we rewrite (68) using (69) which yields

\[ \psi(0) = \frac{1 - \psi(0) p_{00} - \pi (1 - a))/(1 - az^i)}{1 - \psi(0)(1)} \]

Simple modifications to (73) lead to

\[ \psi(0) = \frac{p_{11} \psi(0)}{(1 - az^i)(1 - \psi(0)) + p_{00} \psi(0)} = \frac{p_{11} \psi(0)}{(1 - \pi a) + (p_{00} - p_{11} + \pi a) \psi(0)(1)} \]

Using (71), the last equality becomes

\[ \psi(0) = \frac{p_{11}}{p_{00} - \pi (1 - a)} = \frac{a}{1 - a} \frac{a}{1 - q} \frac{q}{1 - a} \]

which corresponds to (70).
4. Alternative algorithm for conditional non-ruin probabilities

In this section, we present two alternative algorithms to the ones proposed in Cossette et al. (2003) to compute the conditional non-ruin probabilities. These algorithms are based on the exact compound geometric tail representation of \( h(u,i) \), \( u \in \mathbb{N} \) for \( i \in [0,1] \) and the application of Panjer’s algorithm (see Panjer, 1981). At first glance, it is not obvious that one can recourse to Panjer’s algorithm to compute the conditional non-ruin probabilities in our context. Our aim here is not to compare the computing efficiency of the different algorithms suggested but rather to show that Panjer’s algorithm can be used. Also, an advantage of these algorithms is that, given the initial occurrence state \( I_0 \), one can calculate the conditional non-ruin probabilities for the desired \( i \) rather than for both \( i \in [0,1] \).

Before stating the desired results, we need the following definitions.

**Definition 4.** For \( i \in [0,1] \), we define \( \xi(u|i) \) as

\[
\xi(u|i) = \begin{cases} 
\phi(0|i), & u = 0, \\
\phi(u|i) - \phi(u-1|i), & u \in \mathbb{N}^*.
\end{cases}
\] (74)

Clearly, \( \xi(u|i), u \in \mathbb{N} \) is a p.m.f. since \( \xi(u|i) \geq 0 \) for \( u \in \mathbb{N} \) and

\[
\sum_{u=0}^{\infty} \xi(u|i) = \phi(0|i) + \sum_{u=1}^{\infty} \phi(u|i) - \phi(u-1|i) = \lim_{u \to \infty} \phi(u|i) = 1.
\]

In Proposition 5, we prove that \( R_i(z) = (1 - z)^i \tilde{\phi}(z|i) \) is the p.g.f. of a compound geometric distribution for both \( i = 0 \) and \( i = 1 \) with

\[
R_i(z) = \frac{1 - \phi(0|i)}{1 - \psi(0|i)} P_i(z),
\] (75)

where \( P_i(z) \) and \( P_i(z) \) are defined, respectively, in (45) and (43). Based on (74), we can rewrite \( R_i(z) \) as

\[
R_i(z) = (1 - z) \tilde{\phi}(z|i) = \left(1 - z\right) \sum_{u=0}^{\infty} \xi(u|i) = \phi(0|i) + \sum_{u=1}^{\infty} z^u (\phi(u|i) - \phi(u-1|i)) = \sum_{u=0}^{\infty} z^u \xi(u|i),
\]

which means that \( R_i(z) \) is the p.g.f. associated to the p.m.f. \( \{\xi(u|i), u \in \mathbb{N}\} \).

In the following propositions, we present an algorithm for the computation of the p.m.f. \( \{\xi(u|i), u \in \mathbb{N}\} \) for \( i = 0 \) and \( i = 1 \) from which the conditional non-ruin probabilities \( \phi(u|i), u \in \mathbb{N} \) follow easily considering (74).

In Proposition 5, we define the p.g.f. \( \tilde{h}(z) \) as \( \tilde{h}(z) = \tilde{f}_{g(z)/z} \). We assume from now on that \( h \) is the p.m.f. associated to \( \tilde{h}(z) \). Similarly, we define, in Proposition 5, \( \tilde{g}(z) \) as

\[
\tilde{g}(z) = \frac{1 - \pi/p_{00}}{1 - (\pi/p_{00}) h(z)},
\] (76)

which is the p.g.f. of a compound geometric distribution of the form (26) with \( \gamma = \pi/p_{00} \) and \( \tilde{q}_i(z) \) is \( \tilde{h}(z) \). \( g \) and \( G \) hold, respectively, for the p.m.f. and the c.d.f. related to \( \tilde{g}(z) \).

**Proposition 9.** The p.m.f. of \( \{\xi(u|i), u \in \mathbb{N}\} \) can be calculated recursively by applying twice Panjer’s algorithm.

- **First step:** apply Panjer’s algorithm to the compound geometric distribution with parameters \( \gamma = \pi/p_{00} \) and \( p_x = 1 - G(x-1), \ x \in \mathbb{N}^+ \).
- **Second step:** apply again Panjer’s algorithm to the compound geometric distribution with parameters \( \gamma = \phi(0|i) \) and \( p_x = p_{00} \) where

\[
p_{00}(x) = \frac{p_{00}}{\pi(0|i)} (1 - G(x-1)), \ x \in \mathbb{N}^+.
\]
Proof. The first step follows from (76). We then obtain the p.m.f. $g$ and its c.d.f. $G$. In Proposition 5, we deduce that $p_0$, given in (47) and which is function of $G$, is the p.m.f. associated to the p.g.f. $P_0(z)$ which appears in (75). Since $\{\xi(u|i), u \in \mathbb{N}\}$ is the p.m.f. associated to the p.g.f. $R(z)$, this completes the proof of the second step.

Proposition 10. The p.m.f. $\{\xi(u|1), u \in \mathbb{N}\}$ can be calculated recursively by applying Panjer's algorithm to the compound geometric distribution with parameters $\gamma = \psi(0|1)$ and $p_X = p_1$ where $p_1$ is given by

$$p_1(x) = p_{01}(1 - F_B(x)) + \pi f_B(x + 1) \frac{\psi(0|1)(p_00 - \pi f_B(1))}{\psi(0|1)(p_00 - \pi f_B(1))}, \quad x \in \mathbb{N}^+.$$ 

Proof. Since $p_1$, given in (44), is the p.m.f. associated to the p.g.f. $P_1(z)$ and $\{\xi(u|i), u \in \mathbb{N}\}$ is the p.m.f. associated to the p.g.f. $R(z)$, the result easily follows.

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