Compound binomial risk model in a markovian environment

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Abstract

In this paper, we propose a compound binomial model defined in a markovian environment which is an extension to the compound binomial model proposed by Gerber (1988a,b) (Mathematical fun with the compound binomial process, ASTIN Bull. 18, 109–123; Mathematical fun with ruin theory, Ins. Math. Econ. 7, 15–23). An algorithm is presented for the computation of the aggregate claim amount distribution for a fixed time period. We focus on infinite-time ruin probabilities and propose a numerical algorithm to compute their numerical values. Along the same lines as Gerber’s compound binomial model which can be used as an approximation to the classical risk model, we will see that the compound binomial model defined in a markovian environment can approximate the risk model based on a particular Cox model, the marked Markov modulated Poisson process. Finally, we compare via stochastic ordering theory our proposed model to two other risk models: Gerber’s compound binomial model and a mixed compound binomial model. Numerical examples are provided.

Keywords: Dependent Bernoulli; Markovian environment; Compound binomial model; Cox processes; Ruin probabilities; Upper bounds; Stochastic ordering

1. Introduction

In this paper, we propose a compound binomial model defined in a markovian environment as an extension to the compound binomial model proposed by Gerber (1988a,b). One of the interesting features of Gerber’s model is that it can be used as a proxy to the classical compound Poisson risk model (see Dickson et al., 1995; De Vylder and Marceau, 1996). Moreover, since, in Gerber’s model, there exist simple recursive formulas
to compute the aggregate claim amount distribution and the ruin probabilities, risk measures under the classical risk model can be approximated by the corresponding ones obtained under the compound binomial model. Along the same lines, we will illustrate that the continuous-time risk model based on a marked Markov modulated Poisson process, a particular Cox process, can be approximated by a compound binomial model defined in a markovian environment. The interest in such an approximation method is two-fold: first, quantities of interest, including ruin probabilities, cannot, in most cases, be exactly evaluated under the risk model based on the marked Markov modulated Poisson process. Second, in our discrete-time risk model, the same quantities of interest are usually easier to compute and can then serve as approximations to those in the continuous-time risk model.

Other methods have been proposed in the actuarial literature to approximate the risk measures in a risk model based on a marked Markov modulated Poisson process. Among the proposed alternatives, simulation procedures and approximation methods (e.g. diffusion method), which rely both on complex mathematical tools, are used to approximate ruin probabilities (see Asmussen, 1989; Rolski et al., 1999 for more details). Compared to such methods, the compound binomial model defined in a markovian environment can be seen as a simple and efficient alternative. It provides accurate results and relies mainly on simple recursive algorithms to approximate risk measures in the continuous-time risk model.

We begin with a brief description of the compound binomial model in order to introduce our proposed extension. In the compound binomial model, time is measured in discrete units $0, 1, 2, \ldots$ and at most one claim can occur in each time period. The total claim amount over the first $k$ periods, $S_k$, is defined as $S_k = Y_1 + \cdots + Y_k$ where $Y_j$ corresponds to the claim amount in the $j$th period.

It is assumed that $I = \{I_k, k \in \mathbb{N}\}$ is a sequence of i.i.d. Bernoulli r.v.'s and $B = \{B_k, k \in \mathbb{N}\}$ is a sequence of strictly positive i.i.d. r.v.'s. Both sequences are assumed independent of one another which implies that the r.v.'s $Y_j$ are also i.i.d.

In the compound binomial model defined in a markovian environment, the r.v.'s $Y_j$ are still identically distributed but no longer independent. Indeed, the claim occurrence process $I$ and the claim amount process $B$ are both defined in function of an underlying Markov chain $\Theta = \{\Theta_k, k \in \mathbb{N}\}$ defined over a finite state space $\{\theta^{(1)}, \ldots, \theta^{(m)}\}$. The markovian process $\Theta$ may represent the evolution from one period to the subsequent of the economic or the environmental and climatic conditions in a specific area. The Markov chain $\Theta$ is assumed homogeneous, irreducible and ergodic with transition probability matrix

$$
\Gamma = \begin{pmatrix}
\gamma_{11} & \cdots & \gamma_{1m} \\
\vdots & \ddots & \vdots \\
\gamma_{m1} & \cdots & \gamma_{mm}
\end{pmatrix},
$$

(2)

where $\gamma_{ij} = \Pr(\Theta_{k+1} = \theta^{(j)} | \Theta_k = \theta^{(i)})$ for $k \in \mathbb{N}$ and $i, j \in \{1, \ldots, m\}$. The stationary probabilities associated to $\Gamma$ are $\xi = (\xi_1, \ldots, \xi_m)$. We also assume that the initial probabilities of the process $\Theta$ are the stationary probabilities. As in the compound binomial model, we assume that at most one claim can occur in each period. The probability of a claim occurrence in a given period depends on the state of the stochastic process $\Theta$ at that time. Given that $\theta_{i+1} = \theta^{(j)}$, the occurrence r.v. $I_j$ is Bernoulli distributed with mean $\alpha_{ij}$ ($\alpha_{ij} \in [0, 1]$) for $i, j \in \{1, \ldots, m\}$. Without loss of generality, we assume that the occurrence probabilities $\alpha_j = (\alpha_{1j}, \ldots, \alpha_{mj})$ are such that $\alpha_1 \leq \cdots \leq \alpha_m$.

$$
\alpha_1 \leq \cdots \leq \alpha_m,
$$

(3)
Given $I$, the occurrence r.v.'s $I_k$ are mutually independent. In this case, $I$ is referred to a Bernoulli process defined in a markovian environment.

In our extension, we further assume that, if a claim occurs, the claim amount is function of the state of the markovian environment. More precisely, we assume that, given $I_j = 1$ and that a claim occurs at time $j$, the strictly positive discrete claim amount $c_j$ has p.m.f. $f_{B_j}$, p.g.f. $f_{B_j}$, m.g.f. $M_{B_j}$, mean $\mu_{B_j}$ and variance $\sigma_{B_j}^2$. Given $I$, the claim amount r.v.'s $(B_j, j \in \mathbb{N})$ are mutually independent and are also independent of the claim occurrence process $I$.

Let $Y = \{S_k, k \in \mathbb{N}^+\}$ be the aggregate claim amount process where $S_k$, representing the total claim amount over the first $k$ periods, is defined as $S_k = Y_1 + \cdots + Y_k$. The r.v. $Y_j$ holds for the eventual claim amount in period $j$ and is defined as in (1). The surplus process $U = \{U_k, k \in \mathbb{N}\}$ is defined as $U_0 = u$ and $U_k = u + c_k - S_k$ for $k \in \mathbb{N}^+$ where $u (u \in \mathbb{N})$ is the initial surplus level and $c (= 1)$ is the unit premium rate received per period. It is further assumed that the expected claim amount over a single period is strictly inferior to the unit premium, i.e.

$$\sum_{i=1}^{m} \xi_i \alpha_i \mu_B < 1. \quad (4)$$

Note that the compound binomial model defined in a markovian environment differs from discrete-time models defined in a markovian environment studied, e.g. by Lehtonen and Nyrhinen (1992), Nyrhinen (1998), Lillo and Semeraro (2004). In their models, the r.v.'s $Y_k$ correspond to aggregate claims over a single period such as a year or a semester. In the compound binomial model defined in a markovian environment however, the periods are of smaller length and at most one claim can occur in each period. Also, note that other extensions to the compound binomial model have already been proposed in the actuarial literature (see e.g. Yuen and Guo, 2001; Cossette et al., 2003).

The present paper is constructed as follows: in Section 2, we first examine the aggregate claim amount process in the compound binomial model defined in a markovian environment. In Section 3, we provide recursive formulas and a numerical algorithm to compute respectively the finite-time and infinite-time non-ruin probabilities under the framework of this extension. In Section 4, we use the compound binomial model defined in a markovian environment to approximate a risk model based on a marked Markov modulated Poisson process. Next, we compare, via stochastic ordering theory, our extended model to two other discrete-time risk models: the compound binomial model and a mixed compound binomial model. Finally, we conclude with two upper bounds for the infinite-time ruin probabilities in the compound binomial model defined in a markovian environment.

2. Compound binomial model in a markovian environment

Let us recall that $S_k = Y_1 + \cdots + Y_k$ corresponds to the aggregate claim amount over the first $k$ periods. In the present paper, we refer to $Y = \{S_k, k \in \mathbb{N}\}$ as a compound binomial process defined in a markovian environment when the claim occurrence process $I$ is a Bernoulli process in a markovian environment.

First, one easily deduces that the expectation and the variance of $Y_1$ are respectively given by $E[Y_1] = \sum_{i=1}^{m} \xi_i \alpha_i \mu_B$ and $\text{Var}(Y_1) = \sum_{i=1}^{m} \xi_i \alpha_i (\sigma_B^2 + (\mu_B)^2) - (E[Y_1])^2$ for $j \in \mathbb{N}^+$. It then follows that $E[S_k] = k E[Y_1]$ and

$$\text{Var}(S_k) = k \text{Var}(Y_1) + 2 \sum_{i=1}^{k-1} k \sum_{j=i+1}^{k} \left[ \sum_{i=1}^{m} \sum_{l=1}^{m} \xi_i \xi_j \alpha_i \alpha_j \mu_B ^2 \mu_B - E[Y_1]^2 \right].$$
Proposition 1. In the compound binomial model defined in a markovian environment, the conditional p.m.f. of $S_k$ ($k \in \mathbb{N}^+$) can be computed recursively with

$$p_{Sk}(j) = \frac{\sum_{l=1}^{m} \alpha_l (1 - a_l) p_{Sk-l}(0) \phi_l}{\sum_{i=1}^{N} (1 - a_i) p_{Sk-i}(j) \phi_i + \alpha \sum_{l=1}^{m} \beta_l (1 - a_l) p_{Sk-l}(j) \phi_l}, \quad j \in \mathbb{N}^+$$

Proof. The case $j = 0$ is obvious. For the case $j \in \mathbb{N}^+$, we first condition on the r.v.’s $\Theta_k$, $I_k$ and $B_k$. Since $(F_{i+k} | \Theta_k = \theta_i^{(k)})$ is distributed as $(F_{i+k} | \Theta_k = \theta_i^{(k)})$ for $I \sim \Gamma$ and $k \in \mathbb{N}$ and given the stationarity of $\Theta_k$, the result follows.

A numerical example is provided in Section 5 to illustrate the computation of the p.m.f. of $S_k$. Note that the recursive formulas just presented will be useful to approximate the distribution of the aggregate claim amount process in the risk model based on the marked Markov modulated Poisson process.

3. Ruin probabilities

We recall that $U_i = (U_k, k \in \mathbb{N}^+)$ is the surplus process in the compound binomial model defined in a markovian environment which is defined as $U_0 = u$ and

$$U_k = u - W_k = u - \sum_{i=1}^{k} (Y_i - 1),$$

for $k \in \mathbb{N}^+$. Also, we define $\tau = \inf \{k \in \mathbb{N} : U_k < 0\}$ as the time of ruin where $\tau = \infty$ if ruin never occurs.

Let us also define the conditional finite-time ruin and non-ruin probability as $\psi(u,n) = \Pr(\tau \leq n | \Theta_k = \theta_i^{(k)})$ and $\phi(u,n) = 1 - \psi(u,n)$, respectively for $i \in \{1, \ldots, m\}$. The unconditional finite-time ruin and non-ruin probability, $\psi(u,n)$ and $\phi(u,n)$, can be derived respectively from the conditional ones with $\psi(u,n) = \sum_{l=1}^{m} \alpha_l (1 - a_l) \phi_l$ and $\phi(u,n) = 1 - \psi(u,n)$. For convenience, we also define the infinite-time ones by simply letting $n \to \infty$ (and therefore dropping the argument $n$) in our previous definitions of the conditional or unconditional finite-time (non-) ruin probabilities.

We first propose an algorithm to compute ruin probabilities over a finite-time horizon. The remaining of the section will be devoted to infinite-time ruin probabilities.
Proposition 2. In the compound binomial model defined in a markovian environment, the conditional finite-time non-ruin probabilities satisfy the following recursive formula

\[
\begin{pmatrix}
\phi(u-1, n+1) \\
\vdots \\
\phi(u-1, n+1)
\end{pmatrix}
= \sum \otimes \left( I - \text{diag}(\alpha) \right) \otimes
\begin{pmatrix}
\sum_{k=1}^{n} \phi(u-k, n) f_{B_{1}}(k) \\
\vdots \\
\sum_{k=1}^{n} \phi(u-k, n) f_{B_{m}}(k)
\end{pmatrix}, \quad (u \in \mathbb{N}, n \in \mathbb{N})
\]

where \( \otimes \) is the matrix product operator, \( \text{diag}(\alpha) \) holds for \( \text{diag}(\alpha(1), \ldots, \alpha(m)) \), \( I \) is the identity matrix and, for \( u \in \mathbb{N}, \phi(u, 0|\theta(j)) = 1 \) for \( j \in \{1, \ldots, m\} \).

Proof. We condition respectively on the r.v.'s \( \Theta_{1}, I_{1} \) and \( B_{1} \) and the result follows from the stationarity of the surplus process \( \{U_{k}, k \in \mathbb{N}\} \). □

In the compound binomial model, infinite-time ruin probabilities can be computed recursively since the starting point, the infinite-time ruin probability with an initial surplus of zero, has a closed-form expression. A recursive formula will still be derived below for the infinite-time ruin probabilities in the compound binomial model defined in a markovian environment but no explicit expression has yet been found for the starting points. Based on that recursive formula, we propose a numerical method to evaluate the infinite-time ruin probabilities. This method also uses a result proven in Proposition 3 that \( \psi(u) \rightarrow 0 \) as \( u \rightarrow \infty \) under some conditions.

Note that Reinhard and Snoussi (2002) have found an explicit expression for the infinite-time ruin probabilities under a discrete semi-Markov risk model which is a special case of our extended model. Even in this simplified model, the expression for the ruin probabilities is not easily obtained.

In order to simplify the proof of Proposition 3, some definitions are required. Let \( V_{i}^{j} = \sum_{k=1}^{i} 1_{\{\Theta_{k} = \theta(j)|A\}} \) be the elapsed time by the stochastic process \( \Theta \) in state \( \theta(j) \) over the first \( k \) periods where \( A \) takes the value 1 if \( A \) is true and 0 otherwise. We also define \( W_{i}^{j} \) as the amount of decrease of the surplus process over the first \( k \) periods when the stochastic process \( \Theta \) is in state \( \theta(j) \), \( W_{i}^{j} = \sum_{k=1}^{i} 1_{\{\Theta_{k} = \theta(j)|A\}} (Y_{j} - 1) \).

Proposition 3. With \( A \) and \( B \) as defined in (4), the infinite-time ruin probability tends to 0 as the initial surplus \( u \) goes to \( \infty \).

Proof. Taking the limit as \( n \rightarrow \infty \) of \( W_{i}/n \) yields

\[
\lim_{n \rightarrow \infty} \frac{W_{i}}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} W_{i}^{j} = \sum_{i=1}^{\infty} \lim_{n \rightarrow \infty} \frac{V_{i}^{j}}{n} W_{i}^{j} = \sum_{i=1}^{\infty} \frac{V_{i}^{j}}{n} W_{i}^{j}.
\]

(5)

Since \( A \) and \( B \) are irreducible and ergodic, it follows that

\[
\lim_{n \rightarrow \infty} \frac{V_{i}^{j}}{n} = \xi_{i}.
\]

(6)
Also, similar as in Rolski et al. (1999) in the continuous case, \( W_n \) is distributed as \( \sum_{i=1}^{L_n} (Y_i - 1) \) where \( Y_i \) has p.m.f. \( (1 - \alpha_i) \delta_0 + \alpha_i f_{\theta_i} \) where \( \delta_0 \) equals 1 if \( Y_i = 0 \) and 0 otherwise. Since the Markov chain \( \{ \Theta \} \) is irreducible, \( \lim_{n \to \infty} W_n = \infty \) and

\[
\lim_{n \to \infty} \frac{W_n}{V_n} = \lim_{n \to \infty} \frac{\sum_{i=1}^{L_n} (Y_i - 1)}{V_n} = \lim_{n \to \infty} \frac{\sum_{i=1}^{L_n} (Y_i - 1)}{n} = \lim_{n \to \infty} L_n/n,
\]

where \( L_n = \sum_{i=1}^{L_n} (Y_i - 1) \). Therefore, \( \{ L_n \} \) is a random walk for \( \ell \in \{ 1, \ldots, m \} \) and, from the strong law of large numbers, one finds

\[
\lim_{n \to \infty} L_n/n = \mathbb{E}[Y_1 - 1] = \alpha \mu \phi(1) - 1,
\]

for \( \ell \in \{ 1, \ldots, m \} \).

By combining (7) and (6) to (5), we obtain \( \lim_{n \to \infty} \mathbb{E}[W_n/n] = \sum_{i=1}^{m} \xi_i \alpha_i \mu \phi(1) - 1 \) and, from (4), this implies \( \lim_{n \to \infty} W_n = -\infty \) and thus ensures that \( \max_{n \in \mathbb{N}} W_n \) is finite a.s. Consequently, \( \lim_{n \to \infty} \psi(u) = \lim_{n \to \infty} \mathbb{P}(\max_{n \in \mathbb{N}} W_n > u) = 0 \).

Since, from Proposition 3, \( \lim_{n \to \infty} \psi(u) = 0 \), the idea of the numerical algorithm that we propose to compute the infinite-time non-run probabilities is to let the conditional infinite-time non-run probabilities \( \phi(u)^{\theta(i)} \) equal 1 for \( u = n, n + 1, \ldots \) where \( n \) is sufficiently large. The larger \( n \) is fixed, the more accurate the non-run probabilities \( \{ \phi(u)^{\theta(i)}, u \in \mathbb{N} \} \) will be, at the same time, the more time-consuming the numerical method will be. An interesting approach to determine a reasonable and adequate value for \( n \) is to fix \( n \) as the smallest integer for which upper bounds for the non-conditional infinite-time ruin probabilities given the initial surplus level is smaller than \( \varepsilon \) (where \( \varepsilon \) is small). The value of \( \varepsilon \) chosen is inversely proportional to the level of precision required.

**Proposition 4.** In the compound binomial model defined in a markovian environment, the numerical algorithm proposed to obtain the conditional infinite-time non-run probabilities is

- Fix \( \phi(u)^{\theta(i)} = 1 \) for \( u = n, n + 1, \ldots \) and \( i = 1, \ldots, m \).
- Find \( \phi(u)^{\theta(i)} \) for \( u = 0, \ldots, n - 1 \) and \( i = 1, \ldots, m \) by solving the following system of \( m \times n \) equations \( m \times n \) unknown parameters

\[
\phi(u)^{\theta(i)} = \sum_{j=1}^{n} \gamma(j) \left[ (1 - \alpha) \phi(u + 1)^{\theta(i)} + \alpha \sum_{j=1}^{n} \phi(u + 1 - j)^{\theta(i)} f_{\theta(i)}(j) \right]
\]

**Proof.** First, by conditioning respectively on the r.v.’s \( \Theta_1, I_1 \) and \( B_1 \) and from the stationarity of the surplus process \( \{ E_k, k \in \mathbb{N} \} \), one finds

\[
\phi(u)^{\theta(i)} = \sum_{j=1}^{n} \gamma(j) (1 - \alpha) \phi(u + 1)^{\theta(i)} + \sum_{j=1}^{n} \gamma(j) \phi(u + 1 - j)^{\theta(i)} f_{\theta(i)}(j)
\]

for \( i \in \{ 1, \ldots, m \} \) and \( u \in \mathbb{N} \). Given that \( \phi(u)^{\theta(i)} = 1 \) for \( u = n, n + 1, \ldots \) and \( i = 1, \ldots, m \), we must solve the system of \( m \times n \) equations \( m \times n \) unknown parameters given by (8) for \( i \in \{ 1, \ldots, m \} \) and for \( u \in \{ 0, \ldots, n - 1 \} \).
In Proposition 4, we choose to directly solve the system of equations to obtain the desired probabilities contrarily to Reinhard and Snoussi (2004) who, in a similar context, resort to a monotonically converging algorithm. Note that a simulation procedure inspired by Asmussen (1989) in a continuous-time setting could also have been used.

4. Application to a Cox process

4.1. Risk model based on a Cox process

In the classical risk model, it is assumed that the claim arrival process is a Poisson process. This assumption implies a constant claim arrival intensity and, in various situations, such an assumption is inadequate. In such situations, Cox processes can be used as interesting alternatives since the Cox process, also called doubly stochastic Poisson process, has a stochastic claim arrival intensity rate. A general treatment of Cox processes is presented in e.g. Cox (1955), Brémaud (1981), Daley and Vere-Jones (1988). Their applications in risk theory and ruin theory can be found in e.g. Asmussen (1989), Grandell (1991, 1997) and Rolski et al. (1999). We consider a special case of Cox processes, the marked Markov modulated Poisson process for which we give a brief description (see also example 4 of 12.2.2 in Rolski et al., 1999).

In Proposition 4, we choose to directly solve the system of equations to obtain the desired probabilities contrarily to Reinhard and Snoussi (2004) who, in a similar context, resort to a monotonically converging algorithm. Note that a simulation procedure inspired by Asmussen (1989) in a continuous-time setting could also have been used.

For risk models based on Cox processes, interest have been mainly devoted to the problem of computing the finite-time and infinite-time ruin probabilities (see e.g. Janssen and Reinhard, 1985; Reinhard, 1984; Asmussen, 1989; Rolski et al., 1999). For a fixed \( t > 0 \), one can also be interested in the behavior of \( S(t) \) via its c.d.f. \( F_{\xi(t)} \) and...
its stop-loss premium $\pi_{\delta\alpha}$. As mentioned previously, one can resort to simulation methods to approximate these risk measures in the risk model based on a marked Markov modulated Poisson process. In this paper, however, we rely on the compound binomial model defined in a markovian environment.

4.2. Approximation of a Cox process by the compound binomial model defined in a markovian environment

As mentioned in Section 1, Gerber's compound binomial model has been used to approximate the classical compound Poisson risk model. Along the same lines, we use the compound binomial model defined in a markovian environment to approximate the risk model based on a marked Markov modulated Poisson process. This approximation is performed by discretizing the continuous-time risk model and rescaling the resulting discrete-time risk model to fall back on a compound binomial model defined in a markovian environment. Note that, smaller is the discretization pace, more accurate is the resulting approximation. So, let us divide in small intervals of length $\delta$ the time horizon (finite or infinite) considered in the continuous-time risk model. Note that, in the finite-time case the time horizon $\delta$ must be chosen such that $3\delta \in \mathbb{N}^+$: $n\delta = t$. For convenience, we define by $\eta$ the number of periods of length $\delta$ in the desired finite-time horizon.

So, we denote by $\hat{P}(\delta)$ the transition probability matrix over a period of length $\delta$ associated to the intensity matrix $Q$ of the continuous-time Markov chain $\bar{\theta}$

$$
\hat{P}(\delta) = \begin{pmatrix}
p_{11}(\delta) & \cdots & p_{1n}(\delta) \\
\vdots & \ddots & \vdots \\
p_{m1}(\delta) & \cdots & p_{mn}(\delta)
\end{pmatrix},
$$

where $p_{ij}(\delta) = \Pr(\theta_{j+1} = \theta^0 | \theta_j = \theta^0)$, corresponds to the probability of moving from state $\theta^0$ at time $t$ to state $\theta^0$ at time $t + \delta$ for $i, j \in [1, \ldots, m]$. The procedure used to compute $\hat{P}(\delta)$ can be found, for example, in theorem 8.1.4 of Rolski et al. (1999) and is briefly recalled here for completeness. First, we express a diagonalizable and square matrix $\bar{Q}$ as $\bar{Q} = \bar{C} \otimes \bar{D} \otimes \bar{C}^{-1}$ where $\bar{D}$ is a diagonal matrix with elements $d_1, \ldots, d_n$ corresponding to the eigenvalues of $\bar{Q}$, and the columns $\bar{C}_i$ of the matrix $\bar{C}$ are the eigenvectors associated to the eigenvalue $d_i$. Therefore, for a fixed $\delta > 0$, one finds that $\bar{P}(\delta) = \bar{C} \otimes \bar{E}(\delta) \otimes \bar{C}^{-1}$ where $\bar{E}(\delta) = \sum_{j=0}^{\eta} (\delta d_i)^j$ is a diagonal matrix with elements $e^{d_1\delta}, \ldots, e^{d_n\delta}$.

Note that, to approximate the risk model based on a marked Markov modulated risk model, (10) is the transition probability matrix of the discrete-time model, i.e. $Q = \hat{P}(\delta)$. Since $\delta$ is chosen relatively small, we also have

$$
\Pr(N(j + 1) = 0 | \delta, \beta) = \Pr(\theta_{j+1} = \theta^0),
$$

for $j = 0, 1, \ldots, \eta - 1$ which permits to obtain the occurrence probabilities $\alpha$ in the compound binomial model defined in a markovian environment used to approximate the continuous-time risk model. Without loss of generality, we assume that $\hat{\alpha} = 1$ in the risk model based on a marked Markov modulated Poisson process. In such a case and with a discretization pace $\delta$ to perform the approximation, the p.m.f. $f_{\delta\alpha}$ in the compound binomial model defined in a markovian environment is given by

$$
f_{\delta\alpha}(k) = \begin{cases} 
F_{\delta\alpha}(\delta k) - F_{\delta\alpha}((k - 1)\delta), & k \in \mathbb{N}^+ \\
0, & \text{otherwise}
\end{cases}.
$$

The rescaling is necessary since, with a discretization pace $\delta$, the premium received over a period of length $\delta$ is $\delta$, whereas in the compound binomial model defined in a markovian environment the premium rate $e$ is always equal to 1.
With the parameters of the discrete-time risk model fixed as described above, we are now able to approximate \( \tilde{S}(t) \), its related risk measures and the finite-time and infinite-time ruin probabilities within the continuous-time risk model by their corresponding ones in the compound binomial model defined in a markovian environment. These approximations are obtained via the recursive formulas of Propositions 1, 2 and 4. In the following example, we examine the performance of this approximation for various values of \( \delta \). We compare our results to those obtained by simulation of the risk model based on a marked Markov modulated Poisson process (see Asmussen, 1989).

**Example 1.** Assume that \( \tilde{\Theta} \) and \( \tilde{\Theta} \) are both defined on a finite state space \( \{\theta(1), \theta(2), \theta(3)\} \). The continuous-time markovian process \( \tilde{\Theta} \) has intensity matrix

\[
Q = \begin{pmatrix}
-0.016 & 0.0128 & 0.0032 \\
0.0400 & -0.0800 & 0.00400 \\
0.0032 & 0.0160 & -0.0192 \\
\end{pmatrix}
\]

with associated stationary probabilities \( \pi = (0.4569, 0.1514, 0.3917) \). The Poisson claim number process in state \( i \) has rate \( \lambda(i) \) where \( (\lambda(1), \lambda(2), \lambda(3)) = (0.008, 0.016, 0.16) \). The claim amount r.v.'s \( \{\tilde{B}_j, j \in \mathbb{N}^+\} \) is a sequence of i.i.d. r.v.'s exponentially distributed with mean 5.625 and the claim amount r.v.'s do not depend on \( \tilde{\Theta} \). Moreover, we assume that \( \tilde{c} \) is equal to 1.

Our objective is to examine the behavior of \( \tilde{S}(50) \) via its c.d.f. and its stop-loss premium and to compute the unconditional infinite-time non-ruin probabilities \( \tilde{\phi}(u) \), \( u \in \mathbb{N} \) in this continuous-time risk model. In order to do so, we use risk measures defined in the compound binomial model defined in a markovian environment as approximations with the following values for \( \delta \): 1, 1/2, 1/4 and 1/8.

In Tables 1–3, one sees, as expected, that when \( \delta \) becomes smaller, the approximation obtained via the compound binomial model defined in a markovian environment becomes better. Ultimately, as \( \delta \to 0 \), the resulting values will correspond exactly to those defined in the risk model based on a marked Markov modulated Poisson process.

<table>
<thead>
<tr>
<th>Our model</th>
<th>Simulation (60,000 simul.)</th>
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<tbody>
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<td>( \delta = 1 )</td>
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<td>10</td>
<td>0.4630</td>
</tr>
<tr>
<td>15</td>
<td>0.5360</td>
</tr>
<tr>
<td>20</td>
<td>0.5960</td>
</tr>
<tr>
<td>25</td>
<td>0.6335</td>
</tr>
<tr>
<td>30</td>
<td>0.7045</td>
</tr>
<tr>
<td>40</td>
<td>0.7960</td>
</tr>
<tr>
<td>50</td>
<td>0.8692</td>
</tr>
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<td>60</td>
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</tr>
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<tr>
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<td>0.9765</td>
</tr>
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<td>100</td>
<td>0.9943</td>
</tr>
<tr>
<td>120</td>
<td>0.9988</td>
</tr>
<tr>
<td>150</td>
<td>0.9999</td>
</tr>
</tbody>
</table>
Table 2  
Approximated values of the stop-loss premium of \( \tilde{S}(50) \)

<table>
<thead>
<tr>
<th>( d )</th>
<th>Our model</th>
<th>Simulation (60,000 simul.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta = 1 )</td>
<td>( \delta = 1/2 )</td>
<td>( \delta = 1/4 )</td>
</tr>
<tr>
<td>15</td>
<td>12.2903</td>
<td>11.5582</td>
</tr>
<tr>
<td>20</td>
<td>10.1047</td>
<td>9.4333</td>
</tr>
<tr>
<td>30</td>
<td>6.5059</td>
<td>6.0529</td>
</tr>
<tr>
<td>35</td>
<td>5.1899</td>
<td>4.7522</td>
</tr>
<tr>
<td>40</td>
<td>4.0425</td>
<td>3.6806</td>
</tr>
<tr>
<td>45</td>
<td>3.0447</td>
<td>2.8122</td>
</tr>
<tr>
<td>50</td>
<td>2.3514</td>
<td>2.1204</td>
</tr>
<tr>
<td>60</td>
<td>1.9252</td>
<td>1.6000</td>
</tr>
<tr>
<td>70</td>
<td>0.8776</td>
<td>0.6046</td>
</tr>
<tr>
<td>80</td>
<td>0.5779</td>
<td>0.3613</td>
</tr>
<tr>
<td>100</td>
<td>0.0739</td>
<td>0.0663</td>
</tr>
<tr>
<td>120</td>
<td>0.0139</td>
<td>0.0127</td>
</tr>
<tr>
<td>150</td>
<td>0.0039</td>
<td>0.0039</td>
</tr>
</tbody>
</table>

The above example illustrates the strengths of using the compound binomial model defined in a markovian environment to approximate the risk model based on a marked Markov modulated Poisson process. It is easy to implement, allows one to decide on its degree of precision and provides accurate results. This makes it an interesting alternative to the use of simulation methods.

5. Compound binomial model and extensions

It seems natural to compare the compound binomial model in a markovian environment to two other risk models: the compound binomial model and a mixed compound binomial risk model, which is another possible extension.
to Gerber’s model. In the compound binomial model, the increments of the aggregate claim amount process are assumed independent. In the other two models, the dependence relation between the corresponding increments is introduced via two different structures. We give below a brief description of the compound binomial model and the mixed compound binomial model.

For the purpose of the comparison, we assume, throughout this section, that the claim amount r.v.’s in the compound binomial model defined in a markovian environment are independent of the markovian process \( \Theta \).

### 5.1. Compound binomial model

In order to make the desired comparison, we define within the compound binomial model the sequence \( \Theta^{ind} = \{\theta^{ind}_k, k \in N^+\} \) of i.i.d. r.v.’s where the marginal distribution of \( \theta^{ind}_k \) is the same as the one of \( \theta \). We assume that the claim occurrence r.v.’s \( \{I_k^{ind}(\theta^{ind}_k = \theta(i)) \sim \text{Bern}(\alpha_i) \text{ for } i = 1, \ldots, m \} \) which yields

\[
\Pr(I_k^{ind} = 1) = \sum_{i \in \Theta} \alpha_i = q.
\]

Clearly, the claim occurrence process \( \{I_k^{ind}, k \in N^+\} \) is a sequence of i.i.d. r.v.’s where \( I_k^{ind} \sim \text{Bern}(q) \). The assumptions regarding the claim amounts \( B_i \) are as stated in Section 1. The claim amounts \( \{Y_k^{ind}, k \in N^+\} \) form a sequence of i.i.d. r.v.’s. The total claim amount process \( \{X_k, k \in N^+\} \) has therefore a compound binomial distribution with parameters \( \theta, q, \) and \( F_\theta \).

### 5.2. Mixed compound binomial model

In the mixed compound binomial model that we consider, we define a discrete mixing r.v. \( A \) which has the state space of the Markov chain \( \Theta \) for support, i.e. \( A \in \{\theta(1), \ldots, \theta(m)\} \). The distribution of \( A \) corresponds to the stationary distribution of \( \Theta \) which represents the markovian environment. Also, we define the sequence of r.v.’s \( \theta^{mix} = \{\theta^{mix}_k, k \in N^+\} \in \Theta^{mix} = A \text{ for } k \in N^+ \) which means that the components of \( \{\theta^{mix}_1, \ldots, \theta^{mix}_m\} \) for \( k = 1, 2, \ldots \) are comonotonic.

Note that, in a mixed compound binomial model, the probability of occurrence depends on a mixing r.v. \( A \) which remains unchanged over time while, in the compound binomial model defined in a markovian environment, the probability of occurrence changes in each period according to an underlying Markov chain \( \Theta \).

Given \( A = \theta(i) \), the occurrence r.v.’s \( I_k^{mix} \) are conditionally independent and Bernoulli distributed with mean \( \alpha_i \). Thus, the claim occurrence process \( \{I_k^{mix}, k \in N^+\} \) is a sequence of dependent and identically distributed Bernoulli r.v.’s with mean \( \sum_{i \in \Theta} \alpha_i = q \). The corresponding total claim amount process \( \{S_k^{mix}, k \in N^+\} \) is given by \( \Pr(S_k^{mix} = j) = \sum_{i \in \Theta} \Pr(S_k^{mix} = j \mid A = \theta(i)) \text{ where } (S_k^{mix} \mid A = \theta(i)) \) has a compound binomial distribution with parameters \( \alpha_i, \alpha_i, \) and \( F_\theta \) for \( k \in N^+ \).

Note that the assumptions previously stated are such that \( I_k^{ind}, I_k \), and \( I_k^{mix} \) have the same marginal distribution which implies that \( Y_k^{ind}, Y_k \), and \( Y_k^{mix} \) also have the same marginal distribution. Clearly, this implies \( E[X_k] = E[S_k^{ind}] = E[S_k^{mix}] \) for \( k \in N^+ \).

### 5.3. Comparison of the three models

To compare the compound binomial model defined in a markovian environment to the compound binomial model and to a mixed compound binomial model, we rely on stochastic ordering theory. Key definitions to make the desired comparison are given below but for more details on ordering of risks concepts, see e.g. Shaked and Shanthikumar (1994), Kaas et al. (1994) or Joe (1997).
Definition 1. Let $\{X_i(\psi), i \in \mathbb{N}^+\}$ be a sequence of r.v.'s where $\psi$ is sequentially stochastically increasing if, for all $n, \{\psi_{n+1} \mid \psi_n = y_n, \ldots, \psi_1 = y_1\}$ is stochastically increasing in $(y_1, \ldots, y_n)$.

Definition 2. Let $\Psi = \{\psi_i, i \in \mathbb{N}^+\}$ be a sequence of r.v.'s. Then, $\Psi$ is sequentially stochastically increasing if, for all $n, \{\psi_{n+1} \mid \psi_n = y_n, \ldots, \psi_1 = y_1\}$ is stochastically increasing in $(y_1, \ldots, y_n)$.

Proposition 5. Let $\Psi$ be sequentially stochastically increasing and $X_i(\psi)$ be stochastically increasing in $\psi$. Let $\Psi^\text{ind} = \{\psi^\text{ind}_i, i \in \mathbb{N}^+\}$ be a sequence of i.i.d. r.v.'s where, for each $i, \psi^\text{ind}_i$ and $\psi^\text{ind}_i$ have the same marginal distribution. Let $\Psi^\text{com} = \{\psi^\text{com}_i, i \in \mathbb{N}^+\}$ be a sequence of comonotonic r.v.'s such that $\psi^\text{com}_i = \psi_i$. Then,

$$\psi^\text{ind} \leq_{\text{sm}} \psi \leq_{\text{sm}} \psi^\text{com},$$

which completes the proof of (12).

Proof. We have $(\psi_1, \ldots, \psi_n) \leq_{\text{sm}} (\psi^\text{com}_1, \ldots, \psi^\text{com}_n)$ for every $n = 2, 3, \ldots$. Since $\Psi$ is sequentially stochastically increasing, Theorem 3.8 of Meester and Shanthikumar (1993) leads to $(\psi^\text{ind}_1, \ldots, \psi^\text{ind}_n) \leq_{\text{sm}} (\psi_1, \ldots, \psi_n)$, for every $n = 2, 3, \ldots$ which completes the proof of (12).

In the following proposition, we show that, if $\Psi$ is sequentially stochastically increasing, the compound binomial model defined in a markovian environment is bounded below by the compound binomial model and bounded above by the mixture of compound binomial models under the stop-loss order. This implies that for any retention level $d \geq 0$, we have, for $k = 2, 3, \ldots$,

$$\pi_{\Psi}(d) \leq \pi_{\Psi_1}(d) \leq \pi_{\Psi_2}(d).$$

Proposition 6. If $\Psi$ is sequentially stochastically increasing, we have

$$\Sigma^{\text{ind}}_{\Psi} \leq_{\text{sm}} \Sigma \leq_{\text{sm}} \Sigma^\text{com},$$

(15)

$$L^{\text{ind}} \leq_{\text{sm}} L \leq_{\text{sm}} L^\text{com},$$

(16)

$$Y^{\text{ind}} \leq_{\text{sm}} Y \leq_{\text{sm}} Y^\text{com},$$

(17)

and

$$S^{\text{ind}} \leq_{\text{sm}} S \leq_{\text{sm}} S^\text{com},$$

(18)

for $k = 2, 3, \ldots$.

Proof. Let $\Psi^\text{ind}, \Psi$ and $\Psi^\text{com}$ correspond to $\Sigma^{\text{ind}}, \Sigma$ and $\Sigma^\text{com}$, respectively. Let also $X(\psi^\text{ind}), X(\psi)$ and $X(\psi^\text{com})$ correspond to $L^\text{ind}, L$ and $L^\text{com}$ which are stochastically increasing because of (3). Then, the results (15) and (16) follow respectively from (12) and (13). Proposition 2 in Denuit et al. (2002) with (16) leads to (17). With (17), we obtain (18).

Note that, since $E(S^{\text{ind}}) = E(S) = E(S^\text{com})$, the stop-loss order in (18) is equivalent to the convex order, i.e.

$$S^{\text{ind}} \leq_{\text{conv}} S \leq_{\text{conv}} S^\text{com}.$$
Table 4
Values of the c.d.f. of $S_{\text{ind}}^{400}$, $S_{\text{mix}}^{400}$ and $S^{400}$

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\Pr(S_{\text{ind}}^{400} \leq j)$</th>
<th>$\Pr(S_{\text{mix}}^{400} \leq j)$</th>
<th>$\Pr(S^{400} \leq j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0000</td>
<td>0.0642</td>
<td>0.0527</td>
</tr>
<tr>
<td>20</td>
<td>0.0001</td>
<td>0.3304</td>
<td>0.2756</td>
</tr>
<tr>
<td>40</td>
<td>0.0016</td>
<td>0.4916</td>
<td>0.4104</td>
</tr>
<tr>
<td>60</td>
<td>0.0113</td>
<td>0.5647</td>
<td>0.4751</td>
</tr>
<tr>
<td>80</td>
<td>0.0431</td>
<td>0.6080</td>
<td>0.5077</td>
</tr>
<tr>
<td>100</td>
<td>0.1113</td>
<td>0.6069</td>
<td>0.5455</td>
</tr>
<tr>
<td>120</td>
<td>0.2208</td>
<td>0.6080</td>
<td>0.5776</td>
</tr>
<tr>
<td>140</td>
<td>0.3612</td>
<td>0.6080</td>
<td>0.6109</td>
</tr>
<tr>
<td>160</td>
<td>0.5124</td>
<td>0.6085</td>
<td>0.6292</td>
</tr>
<tr>
<td>180</td>
<td>0.7696</td>
<td>0.6085</td>
<td>0.6721</td>
</tr>
<tr>
<td>200</td>
<td>0.9532</td>
<td>0.6133</td>
<td>0.7270</td>
</tr>
<tr>
<td>220</td>
<td>0.9981</td>
<td>0.6133</td>
<td>0.8895</td>
</tr>
<tr>
<td>240</td>
<td>1.0000</td>
<td>0.6208</td>
<td>0.9475</td>
</tr>
<tr>
<td>260</td>
<td>1.0000</td>
<td>0.6208</td>
<td>0.9796</td>
</tr>
</tbody>
</table>

Example 2. We consider a markovian environment $\Theta$ defined on the finite-state space $\{\theta^{(1)}, \theta^{(2)}, \theta^{(3)}\}$ with transition probability matrix

$$
\begin{pmatrix}
0.9990 & 0.0008 & 0.0002 \\
0.0025 & 0.9950 & 0.0025 \\
0.0002 & 0.0010 & 0.9988
\end{pmatrix}
$$

and stationary probabilities $\xi = (0.4569, 0.1514, 0.3917)$. In this case, $\Theta$ is sequentially stochastically increasing. Let $\alpha = (0.005, 0.01, 0.1)$ which means that the claim occurrence r.v.'s $I_k$, $I_{\text{mix}}^k$ and $I_{\text{ind}}^k$ are all Bernoulli distributed with mean 0.0430. For the three risk models, the claim amount r.v.'s $\{B_k, k \in \mathbb{N}\}$ are i.i.d. and have a geometric distribution with $\Pr(B_j = j) = (2/19)^{j-1}$ for $j \in \mathbb{N}^+$. In Tables 4 and 5 respectively, we compare the c.d.f. and the stop-loss premium of the aggregate claim amount r.v.'s $S_{\text{ind}}^{400}$, $S_{\text{mix}}^{400}$ and $S^{400}$ which have all mean 163.26.

Note that, in Table 5, for any fixed value $d \geq 0$,

$$
\pi_{\text{ind}}^{400}(d) \leq \pi_{\text{mix}}^{400}(d) \leq \pi_{\text{ind}}^{400}(d).
$$

Table 6 contains the values of $\psi_{\text{ind}}(u)$, $\psi_{\text{mix}}(u)$ and $\psi(u)$ which are respectively the the infinite-time ruin probabilities in the compound binomial model, the mixed compound binomial model and the compound binomial model defined in a markovian environment. Interestingly, as for the stop-loss premium, we notice that, for any initial surplus level $u (u \in \mathbb{N})$,

$$
\psi_{\text{ind}}^{400}(u) \leq \psi(u) \leq \psi_{\text{mix}}^{400}(u).
$$

However, we cannot rely on Proposition 6 to prove this result.

Rather than directly compare the infinite-time ruin probabilities, we will compare the Lundberg coefficients defined as in Müller and Pflug (2001). First, we define $c_{\text{ind}}$, $c_0$ and $c_{\text{mix}}$ as $c_{\text{ind}}(u) = (\log E[e^{u(S_{\text{ind}}^{400} - u)}]/u$, $c_0(u) = (\log E[e^{u(S_{\text{ind}}^{400} - u)}]/u$ and $c_{\text{mix}}(u) = (\log E[e^{u(S_{\text{ind}}^{400} - u)}]/u$. Let $c_{\text{ind}}$, $c_0$ and $c_{\text{mix}}$ correspond to the limit of $c_{\text{ind}}$, $c_0$ and
Table 5
Values of $\pi_{\text{ind}}(d)$, $\pi_{\text{mix}}(d)$ and $\pi_{\text{sum}}(d)$

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\pi_{\text{ind}}(d)$</th>
<th>$\pi_{\text{mix}}(d)$</th>
<th>$\pi_{\text{sum}}(d)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>163.2611</td>
<td>163.2611</td>
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<tr>
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<td>147.1405</td>
<td>146.5292</td>
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<td>135.5163</td>
<td>133.4748</td>
</tr>
<tr>
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</tr>
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<td>14.4663</td>
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<td>59.5160</td>
</tr>
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</tr>
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</tr>
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</tr>
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<td>0.0040</td>
<td>9.1875</td>
<td>6.0961</td>
</tr>
<tr>
<td>450</td>
<td>0.0004</td>
<td>3.7201</td>
<td>2.6153</td>
</tr>
<tr>
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<td>0.0000</td>
<td>1.2658</td>
<td>0.8746</td>
</tr>
<tr>
<td>600</td>
<td>0.0000</td>
<td>0.0896</td>
<td>0.0604</td>
</tr>
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</table>

$c_{\text{ind}}^{\text{mix}}$ when $n \to \infty$ (if the limit exists). We define $\gamma_{\text{ind}}$, $\gamma$ and $\gamma_{\text{mix}}$ respectively as a strictly positive solution to $c_{\text{ind}}(z)=0$, $c(z)=0$ and $c_{\text{mix}}(z)=0$. In Theorem 2.1 of Müller and Pflug (2001), it is shown that

$$\lim_{u \to \infty} \frac{-\ln(\psi_{\text{ind}}(u))}{u} = \gamma_{\text{ind}},$$

$$\lim_{u \to \infty} \frac{-\ln(\psi(u))}{u} = \gamma.$$
and 
\[
\lim_{u \to 0^+} \frac{-\ln[\gamma^\text{mix}(u)\theta]}{\theta} = \gamma^\text{mix}.
\]

In the following proposition, we show that, under some conditions, the Lundberg coefficients \(\gamma^\text{ind}, \gamma\) and \(\gamma^\text{mix}\) are unique and order such as \(\gamma^\text{mix} \leq \gamma \leq \gamma^\text{ind}\). As suggested by Müller and Pflug (2001), this implies that there exists a \(u_0\) such that \(\gamma^\text{ind}(u) \leq \psi(u) \leq \gamma^\text{mix}(u)\) for \(u > u_0\).

**Proposition 7.** If \(\Theta\) is sequentially stochastically increasing and if the m.g.f. \(M_B(u)\) exist for \(u > 0\), then the Lundberg coefficients \(\gamma^\text{ind}, \gamma\) and \(\gamma^\text{mix}\) are unique and order such as

\[
\gamma^\text{mix} \leq \gamma \leq \gamma^\text{ind}.
\]

**Proof.** From Theorem 3.1. in Müller and Pflug (2001) and since (19) is proven in our setting, it only remains to prove that there exists a unique \(\gamma^\text{ind}\) and \(\gamma^\text{mix}\) in \((0, \infty)\) such that \(\gamma^\text{ind}(\gamma^\text{ind}) = \gamma = \gamma^\text{mix}(\gamma^\text{mix}) = 0\).

In that purpose, we examine the behavior of \(c^\text{ind}(u), c(u)\) and \(c^\text{mix}(u)\) for \(u \in \mathbb{R}^+\). First, note that

\[
c^\text{ind}(0) = c(0) = c^\text{mix}(0) = 0.
\]

Also, from (19), one easily deduces that

\[
c^\text{ind}(u) \leq c(u) \leq c^\text{mix}(u),
\]

for \(u \in \mathbb{R}^+\).

In the independent case, one finds that \(c^\text{ind}(u) = \log(1 - q + qM_B(u)) - u\) from which one deduces that

\[
\frac{d}{du} c^\text{ind}(u) \bigg|_{u=0^+} = q\mu - 1 < 0.
\]

For the mixed case, we have

\[
c^\text{mix}_n(u) = \frac{\log\sum_{k=1}^{n-i} \xi_k \epsilon_k^{(n)}(1 - \alpha_k + a_kM_B(u))^\theta}{n}
\]

Deriving \(c^\text{mix}_n(u)\) by \(u\) and taking the limit \(u \to 0^+\), one finds

\[
\frac{d}{du} c^\text{mix}_n(u) \bigg|_{u=0^+} = \sum_{k=1}^{n-i} \frac{\xi_k \epsilon_k^{(n)}(1 - \alpha_k + a_kM_B(u))^\theta}{n} \left(1 - \frac{a_kM_B(u)}{\alpha_k - M_B(u)}\right) < 0,
\]

for any \(n \in \mathbb{N}^+\). So, letting \(n \to \infty\) in (26), one deduces that

\[
\frac{d}{du} c^\text{mix}_n(u) \bigg|_{u=0^+} < 0.
\]

Combining (23), (24), (25) and (27), one also deduces that 

\[
\gamma'(0) < 0.
\]

Consequently, \(c^\text{ind}(u), c(u)\) and \(c^\text{mix}(u)\) are all decreasing functions at \(u = 0\). Moreover, since \(c^\text{ind}(u), c(u)\) and \(c^\text{mix}(u)\) are convex functions, since, in the independent case, it is well known that there exists a unique \(\gamma^\text{ind}\) in \((0, \infty)\) such that \(c^\text{ind}(u) = 0\) when \(M_B(u)\) exists for \(u > 0\) and from (24), the result follows. \(\square\)

In the context of Example 2, we find that \(\gamma^\text{mix} = 0.0059, \gamma = 0.0138\) and \(\gamma^\text{ind} = 0.0073\) which is in line with (22).
6. Upper bounds for the infinite-time ruin probabilities

In this section, we derive upper bounds for the infinite-time ruin probabilities in the compound binomial model defined in a Markovian environment. The two upper bounds are both derived from martingale theory and assume the existence of the m.g.f. \( M_{\theta}(r) \) for \( r > 0 \) and \( i \in \{1, \ldots, m\} \).

6.1. Lundberg-type upper bound

We first derive a simple upper bound for the infinite-time ruin probabilities in our discrete-time risk model similar to the Lundberg exponential bound in the classical risk model. However, an important drawback of this Lundberg-type upper bound is that additional restrictions are needed to ensure its applicability.

Let \( F_n = \sigma(\{Y_k, \Theta_k\}, k = 0, 1, \ldots, n) \) be the \( \sigma \)-field generated by the process \( \{Y_k, \Theta_k\} \) up to time \( n \in \mathbb{N}^+ \).

Using (1), one finds

\[
E[e^{rY_1}|\Theta_0 = \theta(i)] = \sum_{j=1}^{m} \gamma_{ij}(1 - \alpha_j + \alpha_j M_{\theta}(r)).
\]

Also, let \( \{R_i, i = 1, \ldots, m\} \) be the strictly positive real numbers (if they exist) which satisfy

\[
E[e^{rY_1}|\Theta_0 = \theta(i)] = e^r.
\] (28)

Such \( R_i \)'s exist if

\[
E[Y_1|\Theta_0 = \theta(i)] < 1,
\] (29)

which corresponds to the additional restrictions needed to ensure the application of this Lundberg-type upper bound.

In addition, we denote by \( R^* \) the adjustment coefficient defined as

\[ R^* = \min_{i \in \{1, \ldots, m\}} R_i. \]

It is immediate that \( R^* \) is a strictly positive real number if (29) is satisfied for \( \forall i \in \{1, \ldots, m\} \). The adjustment coefficient \( R^* \) can be seen as a crude measure of risk for the surplus process \( \{U_k, k \in \mathbb{N}\} \). The smaller is \( R^* \), the more dangerous is the surplus process \( \{U_k, k \in \mathbb{N}\} \).

Proposition 8. Suppose the existence of \( R^* > 0 \). Upper bounds for the conditional and unconditional infinite-time ruin probabilities are respectively given by

\[
\psi(u|\theta(i)) \leq e^{-R^* u},
\] (30)

and

\[
\psi(u) \leq e^{-R^* u}.
\] (31)

Proof. We obtain (30) by first proving that \( \{e^{-R^*U_k}, k \in \mathbb{N}\} \) corresponds to a supermartingale with respect to \( \{\Theta_k\} \) the filtration \( \{\Theta_k\} \). Since \( e^{-R^*U_k} \) is adapted to \( \{\Theta_k\} \) and \( E[e^{-R^*U_k}] < \infty \), it only remains to prove that \( E[e^{-R^*U_{k+1}}|\Theta_k] \leq e^{-R^* U_k} \). By definition, one knows that

\[
E[e^{-R^*U_{k+1}}|\Theta_k] = E[e^{-R^*(U_{k+1} + Y_{k+1})}|\Theta_k] = e^{-R^* U_k} \frac{E[e^{R^*Y_{k+1}}|\Theta_k]}{e^{R^*}} = e^{-R^* U_k} \frac{E[e^{R^*\theta_{k+1}|\Theta_k}] e^{R^* \theta_k}}{e^{R^*}}.
\] (32)
Since $0 < R^* \leq R$, and due to (29), one deduces that (32) can be rewritten as

$$E[e^{-R^* U_k}|F_k] \leq e^{-R^* u_k},$$

Therefore, it follows that $\{e^{R^* U_k}, k \in \mathbb{N}\}$ is a supermartingale process w.r.t. $\{F_k\}$. Then, from Kolmogorov’s inequality for positive supermartingales (see e.g. Gerber, 1979), one finds

$$\Pr(\max_{k \in \mathbb{N}} \{e^{R^* U_k}\} \geq 1 | \Theta_0 = \theta_0) \leq e^{-R^* u},$$

(33)

for $u \in \mathbb{N}$. However, one knows that

$$\psi(u | \theta_0) = \Pr(\min_{k \in \mathbb{N}} \{U_k\} < 0 | \Theta_0 = \theta_0) = \Pr(\max_{k \in \mathbb{N}} \{e^{R^* U_k}\} > 1 | \Theta_0 = \theta_0).$$

(34)

Combining (33) and (34) result in

$$\psi(u | \theta_0) \leq e^{-R^* u}$$

for $\forall \theta_0 \in \{\theta^{(1)}, \ldots, \theta^{(m)}\}$ which completes the proof of (30).

6.2 Martingale-type upper bound

In this aim of finding a more general upper bound, our goal here is to identify a martingale contrarily to a supermartingale which led to the upper bound proposed in the previous sub-section. Note that the second upper bound overcomes the weakness of the first one since (29) is no longer required in the derivation of this one.

In order to identify a martingale process, let us denote by $M(v)$ the matrix m.g.f. of $\{Y_n - 1, n \in \mathbb{N}\}$

$$M(v) = \begin{bmatrix}
M_{11}(v) & \cdots & M_{1n}(v) \\
\vdots & \ddots & \vdots \\
M_{m1}(v) & \cdots & M_{mn}(v)
\end{bmatrix}$$

for which $M_{ij}(v)$ holds for $E[e^{v(Y_n - 1)} 1_{\{\Theta_n = \theta_i, \Theta_{n+1} = \theta_j\}} | \Theta_0 = \theta_0], \forall n \in \mathbb{N}, V, i, j \in [1, \ldots, m]$. Since $M(v)$ is regular, the highest eigenvalue of $M(v)$ is positive and greater in absolute value than any other eigenvalues of $M(v)$. This specific eigenvalue for the matrix $M(v)$, denoted $\zeta(v)$, is called the spectral radius or the Perron-Frobenius root. Its associated strictly positive right eigenvector, denoted $h(v)$, satisfies

$$M(v) \otimes h(v) = e^{v(\zeta(v))} h(v).$$

(35)

It is based on equality (35) that we will find the martingale process necessary to obtain upper bounds for the infinite-time ruin probabilities.

**Proposition 9.** The process $\{e^{-v U_k}, k \in \mathbb{N}\}$ is a $F_k$-martingale if $v$ is chosen such that

$$\zeta(v) = 0,$$

(36)

where $h(v)$ holds for element in position $j$ of the right eigenvector $h(v)$. 
Proof. Clearly, the process $[e^{-t\Theta_k}h^{(\nu)}(\theta_k)]$ is adapted to the filtration $\{F_t\}$. Next, we prove that

$$E[e^{-t\Theta_k}h^{(\nu)}(\theta_k)|F_k] = e^{-t\Theta_k}h^{(\nu)}(\theta_k) \quad \text{for} \quad k \in \mathbb{N}. $$

One knows that

$$E[e^{-t\Theta_k}h^{(\nu)}(\theta_k)|F_k] = e^{-t\Theta_k}E[e^{(t\Theta_{k+1} - t\Theta_k)h^{(\nu)}(\theta_k)}|F_k] = e^{-t\Theta_k}\sum_{j=1}^{\infty} h^{(\nu)}(j)M_{\theta_k,j}(\theta_k)$$

since $M_{\theta_k,j}(\theta_k) = E[e^{(t\Theta_{k+1} - t\Theta_k)h^{(\nu)}(\theta_k)}|\{\Theta_{k+1} = j\}] = \theta_j^{\nu}$. From (35) and (36), (37) becomes

$$E[e^{-t\Theta_k}h^{(\nu)}(\theta_k)|F_k] = e^{-t\Theta_k}(e^{t\Theta_k}h^{(\nu)}(\theta_k)) = e^{-t\Theta_k}h^{(\nu)}(\theta_k).$$

Finally, one can easily show, from (38), that

$$E[e^{-t\Theta_k}h^{(\nu)}(\theta_k)|F_k] = E[e^{-t\Theta_{k+1}}h^{(\nu)}(\theta_{k+1})|F_k] = E[e^{-t\Theta_k}h^{(\nu)}(\theta_k)] = e^{-t\Theta_k}h^{(\nu)}(\theta_k) < \infty$$

which completes that $[e^{-t\Theta_k}h^{(\nu)}(\theta_k), k \in \mathbb{N}]$ is a $F_t$-martingale. □

Proposition 10. In the compound binomial model defined in a markovian environment, an upper bound to the conditional infinite-time ruin probabilities are given by

$$\psi(u^{(\nu)} \leq \min \left\{ \frac{h^{(\nu)}(u^{(\nu)})}{\min_{i \in \{1,\ldots,n\}} h^{(\nu)}(i)}, e^{-\nu} \right\}. \quad (39)$$

for $\nu > 0$ which satisfies (36).

Proof. One knows, since $\nu > 0$, that the conditional infinite-time ruin probabilities can be written as

$$\psi(u^{(\nu)} = \Pr \left( \max_{k \in \mathbb{N}} U_k < 0 \mid \Theta_0 = \theta_0 \right) = \Pr \left( \max_{k \in \mathbb{N}} \left[ e^{-\Theta_k} \right] > 1 \mid \Theta_0 = \theta_0 \right),$$

from which follows the inequality

$$\psi(u^{(\nu)} \leq \Pr \left( \max_{k \in \mathbb{N}} \left[ e^{-\Theta_k} \right] > \min_{i \in \{1,\ldots,n\}} \min_{j \in \{1,\ldots,m\}} h^{(\nu)}(i)h^{(\nu)}(j) \right) \right).$$

Given that $[e^{-\Theta_k}h^{(\nu)}(\theta_k), k \in \mathbb{N}]$ is a $F_t$-martingale and from Dobb’s inequality for supermartingales, one finds that

$$\psi(u^{(\nu)} \leq \frac{h^{(\nu)}(u^{(\nu)})}{\min_{i \in \{1,\ldots,n\}} h^{(\nu)}(i)} e^{-\nu}.$$ 

Since $h^{(\nu)}(u^{(\nu)})/\min_{i \in \{1,\ldots,n\}} h^{(\nu)}(i)$ can be greater than 1 and $\psi(u^{(\nu)} \leq 1$, (39) is obtained. □
References


