Analysis of the discounted sum of ascending ladder heights

Hélène Cossette, David Landriault, Etienne Marceau *, Khouzeima Moutanabbir

Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Canada
École d’Actuariat, Université Laval, Québec, Canada

Contents lists available at SciVerse ScienceDirect
Insurance: Mathematics and Economics
journal homepage: www.elsevier.com/locate/insmeca

1. Introduction

We consider the Sparre Andersen risk model in ruin theory. Let the claim number process \( N = \{N(t), t \geq 0\} \) be a renewal process where the (positive) interclaim times \( \{W_j, j \in \mathbb{N}^+\} \) form a sequence of independent and identically distributed (iid) rvs with density \( k \), cumulative distribution function (cdf) \( K(t) = \int_0^t \exp(-x) dx \) and Laplace transform (LT) \( L(t) \). Also, the claim amount rvs \( \{X_j, j \in \mathbb{N}^+\} \) are an iid sequence of positive rvs with density \( p \), cdf \( P(x) = 1 - \mathbb{P}(X > x) \), LT \( \mathbb{P} \) and mean \( \mu \). We further assume that any pair \( (W_j, X_j) \) consists of two independent rvs.

Based on the pioneer work of Andersen (1957), we describe the surplus process \( U = \{U(t), t \geq 0\} \) associated to a portfolio of insurance business as

\[
U(t) = u + ct - S(t),
\]

where \( u \geq 0 \) is the initial surplus level, \( c > 0 \) is the premium rate, and \( S = \{S(t), t \geq 0\} \) is the aggregate claim amount process defined as

\[
S(t) = \begin{cases} \sum_{i=1}^{N(t)} X_i, & N(t) > 0, \\ 0, & N(t) = 0, \end{cases}
\]

where \( \delta \geq 0 \), \( w(x, y) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) is the so-called penalty function, and \( I \) is the indicator function (that is \( I(A) = 1 \) if the event \( A \) occurs and 0 otherwise). From the seminal contribution of Gerber and Shiu (1998), a significant body of the ruin theory literature emerged on the analysis of these Gerber–Shiu type functionals in various risk models. The reader is referred to Asmussen and Albrecher (2010, Chapter XII) for a summary on the topic.

Inspired from Trufin et al. (2010), we analyze in Cossette et al. (in preparation) the Value-at-Risk (VaR) and the Tail-Value-at-Risk (TVaR) risk measures which are defined in terms of the maximum aggregate loss \( Z = \sup \{S(t) - ct, t \geq 0\} \), cdf \( F_Z(x) \) and survival function \( F_Z(x) = 1 - F_Z(x) \). The infinite-time ruin probability with an initial surplus \( u \) is

\[
\psi(u) = \Pr (\tau < \infty | U(0) = u) =
\]
The VaR at level $\kappa$, $0 \leq \kappa < 1$, of $Z$ is defined by
\[
\text{VaR}_\kappa (Z) = F_Z^{-1} (\kappa) ,
\]
where $F_Z^{-1} (\kappa) = \inf \{ x \in \mathbb{R} : F_Z (x) \geq \kappa \}$. The TVaR at level $\kappa$, $0 \leq \kappa < 1$, is defined by
\[
\text{TVaR}_\kappa (Z) = \frac{1}{1 - \kappa} \int_0^{\kappa} \text{VaR}_u (Z) \, du
\]
where $E \left[ Z \times 1_{\{Z > 0\}} \right]$ is the truncated expectation of $Z$ and $1_A$ is the indicator function such that $1_A (Z) = 1$, if $Z \in A$, and $1_A (Z) = 0$, if $Z \notin A$. Note that $E \left[ Z \times 1_{\{Z > 0\}} \right]$ can be expressed as $E [Z] - E \left[ Z \times 1_{\{Z \leq 0\}} \right]$. See e.g. Acerbi (2002), Acerbi and Tasche (2002) and McNeil et al. (2005) for details on the VaR and TVaR. As is explained in Cossette et al. (in preparation), either VaR or TVaR of $Z$ can be used to determine the initial surplus $u$, which necessitates the knowledge of the distribution of $Z$.

It is well known that the maximum aggregate loss $Z$ follows a compound geometric distribution, in which the summands consist of the ascending ladder heights. In the present paper, we propose to investigate the distribution of the discounted sum of ascending ladder heights over a finite or an infinite-time intervals. In particular, we derive the expressions for the moments of the discounted sum of ascending ladder heights which may be computed with a moment based approximation using the moments of the discounted sum of ascending ladder heights.

The paper is structured as follows: In Section 2, the discounted sum of the ascending ladder heights is formally defined, and its connection to the discounted aggregate loss in risk theory is established. We also review some (known) results about mixed Erlang distributions which will be useful in the later part of this paper. Section 3 considers the analysis of the moments of the discounted sum of ascending ladder heights in both the classical compound Poisson risk model and the Sparre Andersen risk model with exponential claims. A closed-form expression which involves tractable Erlang densities is identified. In Section 4, a moment-matching method is used to approximate the distribution of the discounted sum of ascending ladder heights over a given time horizon. We apply this approximation to compute the VaR and the TVaR defined in terms of the discounted sum of ascending ladder heights.

2. Preliminaries

We define the net aggregate loss process $Y = \{ Y (t) , \ t \geq 0 \}$ by $Y (t) = S (t) - ct$. Let $\{ \upsilon_i \}_{i=1}^\infty$ be the sequence of ascending ladder height epochs associated to the net aggregate loss process $Y$, i.e.
\[
\upsilon_i = \inf \{ t \geq 0 : Y (t) > 0 \} ,
\]
and recursively
\[
\upsilon_i = \inf \{ t \geq \upsilon_{i-1} : Y (t) - Y (\upsilon_{i-1}) > 0 \} .
\]

For convenience, let $\upsilon_0 = 0$. Also, define $\upsilon_i = \upsilon_i - \upsilon_{i-1}$ to be the duration of the $i$th ladder epoch, and $L_i = Y (\upsilon_i) - Y (\upsilon_{i-1})$ to be its corresponding ascending ladder height (whenever $\upsilon_i < \infty$). Also, let $M$ be the number of ladder epochs of $Y$, i.e. $M = \sup \{ i \in \mathbb{N} : \upsilon_i < \infty \}$. Due to the regenerative property of the process $Y$ at claim instants, it is immediate that $\{(\upsilon_i, L_i)\}_{i=1}^\infty$ forms a sequence of IID random pairs, distributed as $(\tau, |U (\tau)|)$ (i.e. the time to ruin and the deficit at ruin) when the initial surplus level is 0.

It is well known that, in the Sparre Andersen risk model (see e.g., Rolski et al. (1999), Chapter 6), the maximum aggregate loss $Z$ follows a compound geometric distribution, e.g.
\[
Z = \sum_{i=1}^M L_i , \quad M > 0 , \quad \upsilon_0 = 0 , \quad M = 0 ,
\]
where $M$ is a geometric rv with probability mass function (pmf)
\[
Pr (M = m) = (1 - \psi (0)) \psi (0)^m , \quad m = 0, 1, \ldots .
\]

Also, let $Z = \{ Z(t), t \geq 0 \}$ be the maximum aggregate loss process where $Z (t) = \sup \{ Y (s) : 0 \leq s \leq t \}$ or alternatively
\[
Z (t) = \begin{cases} \sum_{i=1}^M L_i \upsilon_i \leq t , & M > 0 , \\ 0 , & M = 0 . \end{cases}
\]

An illustration of a sample path of $Y$ is provided in Fig. 1.

We introduce the concept of time value of money, and extend $Z$ to its discounted version $Z_d = \{ Z_d (t), t \geq 0 \}$ by discounting the ascending ladder heights at a constant force of interest $\delta \geq 0$, i.e.
\[
Z_d (t) = \begin{cases} \sum_{i=1}^M e^{-\delta \upsilon_i} L_i \upsilon_i \leq t , & M > 0 , \\ 0 , & M = 0 . \end{cases}
\]

Note that in risk theory, $Z_d (t)$ can be viewed as the discounted aggregate loss (see, e.g., Lévéillé and Garrido (2001a,b) and Ren (2008)) where the interclaim times and claim sizes are distributed as an arbitrary $\gamma_i$ and $L_i$ respectively. In our context, the pairs $\{(\upsilon_i, L_i)\}_{i=1}^\infty$ are mutually independent, whereas the rvs $\gamma_i$ and $L_i$ are in general not independent (similarly as in Bargès et al. (2009)). This departs from the usual assumption in risk theory for the analysis of the (discounted) aggregate loss.

In the following sections, mixed Erlang distributions will play an important role. We recall definitions and known results on this family of distributions in the following proposition. The reader is primarily referred to Willmot and Woo (2007) and Willmot and Lin (2011) for more details.
Proposition 1. Let $X$ be mixed Erlang with LT $\hat{P}(s) \equiv \int_0^\infty e^{-sx}p(x)\,dx = Q(\beta; \tau)$ where $\beta > 0$, $Q(z) = \sum_{i=1}^{\infty} q_i z^i$ and $\{q_i\}_{i \geq 1}$ is a pmf. Its density is given by

$$p(x) = \sum_{i=1}^{\infty} q_i \tau_{\beta,i}(x),$$

where $\tau_{\beta,i}(x) = \beta^i x^i / i!$ for $x \geq 0$.

(i) The density of $X$ admits the factorization

$$p(x + y) = \sum_{i=1}^{\infty} \kappa_i(y) \tau_{\beta,i}(x),$$

where

$$\kappa_i(y) = \sum_{j=0}^{\infty} q_{i+j} \tau_{\beta,j+i}(y).$$

(ii) The equilibrium distribution of $X$ is also mixed Erlang with density

$$p_{e}(x) = \int^{\infty}_0 \frac{p(y)\,dy}{\mu} = \sum_{j=1}^{\infty} w_j \tau_{\beta,j}(x),$$

where $w_j = \frac{\bar{\tau}_{j}}{\sum_{k=0}^{\infty} \tau_{j,k}}$ and $\bar{\tau}_{j} = \sum_{k=j}^{\infty} \tau_{j,k}$ ($j = 1, 2, \ldots$).

We also state a special case of Eq. (2.6) in Willmot and Woo (2007) which will often be used in the sequel.

Proposition 2. For $0 < \beta_1 \leq \beta_2$,

$$\tau_{\beta_1,t} = \sum_{i=0}^{\infty} q_{i+1} \left( \frac{\beta_1}{\beta_2} \right)^i \tau_{\beta_2,i}(t),$$

where

$$q_{i+1} = \binom{i-1}{k-1} \delta^k (1-\delta)^{i-k}.$$

3. Discounted sum of ladder heights

3.1. General structure

We first examine the LT of the discounted sum of ascending ladder heights over the time interval $[0, t]$. By conditioning on the time and the amount of the first ladder height, one finds

$$E \left[ e^{-Z_2(t)} \right] = Pr(\gamma_1 > t) + E \left[ e^{-e^{-\delta \gamma_1}} \left( L_1 + Z_2(t-\gamma_1) \right) ; \gamma_1 < t \right].$$

where $Z_j(t)$ is identically distributed as $Z_2(t)$ for all $t \geq 0$. The regenerative property of the renewal risk process at claim instants implies that $Z_j(t)$ is mutually independent of either $\gamma_1$ or $L_1$.

By taking the $n$-th derivative ($n = 1, 2, \ldots$) with respect to (wrt) $s$ on both sides of (5) and then letting $s \to 0$, one obtains that

$$E \left[ (Z_2(t))^n \right] = E \left[ e^{-n \delta \gamma_1} \left( L_1 + Z_2(t-\gamma_1) \right) ; \gamma_1 < t \right].$$

Letting $m_{s,n}(t) = E \left[ (Z_2(t))^n \right]$ ($n = 0, 1, 2, \ldots$), simple manipulations of (6) lead to

$$m_{s,n}(t) = \sum_{i=0}^{n} \binom{n}{i} E \left[ e^{-n \delta \gamma_1} \left( L_1 \right)^{n-i} \left( Z_2(t-\gamma_1) \right)^i ; \gamma_1 < t \right]$$

$$= \sum_{i=0}^{n} \binom{n}{i} \int_0^t e^{-n \delta \gamma_1} f_{\gamma_1}(w) \left( L_1 \right)^{n-i-w} \left( Z_2(t-\gamma_1) \right)^i \, dw,$$

where $f_{\gamma_1}$ is the density function of an arbitrary $\gamma_1$. Define $m_{s,n}(t) = \int_0^\infty e^{-s \gamma_1} m_{s,n}(t) \, d\gamma_1$. Taking Laplace transforms on both sides of (7) yields

$$\tilde{m}_{s,n}(t) = \sum_{i=0}^{n} \binom{n}{i} E \left[ e^{-n \delta \gamma_1} \left( L_1 \right)^{n-i} \left( Z_2(t-\gamma_1) \right)^i ; \gamma_1 < t \right] \tilde{m}_{s,i}(t),$$

or alternatively

$$\tilde{m}_{s,n}(t) = \sum_{i=0}^{n} \binom{n}{i} \frac{E \left[ e^{-n \delta \gamma_1} \left( L_1 \right)^{n-i} ; \gamma_1 < t \right]}{1 - E \left[ e^{-n \delta \gamma_1} ; \gamma_1 < t \right]} \tilde{m}_{s,i}(t).$$

(8)

For $\delta > 0$, (8) is also equivalent to

$$\tilde{m}_{s,n}(t) = \sum_{i=0}^{n-1} \binom{n}{i} \frac{E \left[ e^{-n \delta \gamma_1} \left( L_1 \right)^{n-i} ; \gamma_1 < t \right]}{1 - E \left[ e^{-n \delta \gamma_1} ; \gamma_1 < t \right]} \tilde{m}_{s,i}(t).$$

(9)

It is interesting to point out that $\tilde{m}_{s,n}(t)$ is expressed in terms of the discounted moments of the deficit at ruin for an initial surplus of 0 which is known in a variety of risk models (thanks to the numerous advances on the Gerber–Shiu discounted penalty function; see, e.g., Lin and Willmot (2000), Tsai and Willmot (2002), and Ren (2007)).

Remark 3 (Infinite-Time Horizon). From the final value theorem (see Feller (1968)), it is known that

$$\lim_{t \to \infty} \tilde{m}_{s,n}(t) = E \left[ \left( Z_2(t) \right)^n \right],$$

where $Z_2 \equiv Z_2(\infty)$. Hence, pre-multiplying both sides of (9) by $z$ and then taking the limit when $z \to 0$, one arrives at

$$E \left[ \left( Z_2(t) \right)^n \right] = \sum_{i=0}^{n-1} \binom{n}{i} \frac{E \left[ e^{-n \delta \gamma_1} \left( L_1 \right)^{n-i} ; \gamma_1 < \infty \right]}{1 - E \left[ e^{-n \delta \gamma_1} ; \gamma_1 < \infty \right]} E \left[ \left( Z_2(t) \right)^i \right].$$

(10)

We point out that Eq. (10) allows for the recursive calculation of the moments of $Z_2$, the discounted sum of ascending ladder heights, whenever a closed-form expression exists for the discounted moments of the deficit at ruin with an initial surplus of 0.

In a finite-time span, the structure of Eq. (9) indicates that a (recursive) expression for the moments of $Z_2(t)$ can be identified when the inversion of the right-hand side of (9) wrt $z$ can be performed. By further examining (9), one notes that the argument $z$ is present in the Laplace transform argument of the first passage time $\gamma_1$. Capitalizing on the recent advances of the analytic inversion of the Laplace transform of the time to ruin (see e.g. Dickson and Willmot (2005), Borovkov and Dickson (2008) and Landriault et al. (2011)), we will examine in more details two particular Sparre Andersen risk models, namely the compound Poisson risk model and the Sparre Andersen risk model with exponential claims.

3.2. Classical compound Poisson model

The discounted moments of the deficit at ruin have been thoroughly studied in the context of the classical compound Poisson risk model by Lin and Willmot (2000). Capitalizing on their results (notably, Theorem 4.1 on Page 28), we aim at analytically inverting (9) to obtain a closed-form expression for the $n$-th (discounted) moment of the ascending ladder heights in the time horizon $[0, t]$. The following lemma contains a result particularly relevant in this regard.

Lemma 4. In the classical risk model with Poisson arrivals at rate $\lambda > 0$,

$$E \left[ e^{-\delta \gamma_1} \left( L_1 \right)^t ; \gamma_1 < \infty \right] = \int_0^\infty e^{-\delta \gamma_1} g(t) \, dt,$$

(11)
As a by-product, one obtains
\[
\frac{E \left[ e^{-\delta y_1} (L_1)^\lambda; \gamma_1 < \infty \right]}{1 - E \left[ e^{-\delta y_1}; \gamma_1 < \infty \right]} = \int_0^\infty e^{-\delta t} \left\{ \frac{\lambda \mu_{t,1}}{c} P (t) \right\} dt + \int_0^\infty e^{-\delta t} \left\{ \frac{\lambda}{c} \sum_{k=1}^{\infty} \left( \frac{\lambda \mu_k}{c} \right)^k \right\} \mu_{t,1} P (t) dt,
\]
where \( \mu_{n,m} \) is the \( m \)-th moment of the residual lifetime density \( p_t (y) = p (t + y) / \bar{P} (t) \) for \( y \geq 0 \). Letting
\[
f_t (t) = \frac{\lambda}{c} \left( \mu_{t,1} \bar{P} (t) + \sum_{k=1}^{\infty} \left( \frac{\lambda \mu_k}{c} \right)^k \right)
\]
and \( \delta \) is as defined in (14).

it follows that
\[
\frac{E \left[ e^{-\delta y_1} (L_1)^\lambda; \gamma_1 < \infty \right]}{1 - E \left[ e^{-\delta y_1}; \gamma_1 < \infty \right]} = \int_0^\infty e^{-\delta t} f_t (t) \, dt.
\]

Using the Lagrangian identity
\[
e^{-\delta t} = e^{-\frac{\lambda \mu_k}{c} t} + \sum_{k=1}^{\infty} \left( \frac{\lambda \mu_k}{c} \right)^k \int_0^\infty e^{-\frac{\lambda \mu_k}{c} t} \, dw
\]

(see, e.g., Lin and Willmot (1999) and Dickson and Willmot (2005)), one can substitute the LT (15) in \( \rho_b \) into a LT in \( \delta \). Using Eq. (4) of Dickson and Willmot (2005), it is immediate that (15) becomes (11).

Replacing \( \delta \) by \( z + n \delta \) in (11) yields
\[
\frac{E \left[ e^{-\left( z + n \delta \right) y_1} (L_1)^\lambda; \gamma_1 < \infty \right]}{1 - E \left[ e^{-\left( z + n \delta \right) y_1}; \gamma_1 < \infty \right]} = \int_0^\infty e^{-\delta t} \left\{ e^{-n \delta t} \mu_{t,1} P (t) \right\} dt + \int_0^\infty e^{-\delta t} \left\{ \sum_{k=1}^{n-1} \left( \frac{\lambda \mu_k}{c} \right)^k \right\} \mu_{t,1} P (t) dt.
\]

Defining the functional \( \{ \psi_\delta (t) \}, t \geq 0 \) through its LT \( \tilde{\psi}_{n,\delta} (z) \)
\[= z \tilde{\psi}_{n,\delta} (z), \]
and clear that
\[m_{n,\delta} (t) = \int_0^t \psi_{n,\delta} (y) \, dy.
\]

For convenience, we propose to examine the moments of \( Z_\delta (t) \) through the functional \( \{ \varphi_{n,\delta} (t) \}, t \geq 0 \). In this regard, one multiplies both sides of (17) by \( z \) to obtain
\[
\tilde{\varphi}_{n,\delta} (z) = \int_0^\infty e^{-z t} \left\{ e^{-n \delta t} \mu_{t,1} P (t) \right\} dt + \int_0^\infty e^{-z t} \left\{ \sum_{k=1}^{n-1} \left( \frac{\lambda \mu_k}{c} \right)^k \right\} \mu_{t,1} P (t) dt.
\]

By the uniqueness property of Laplace transforms (see, e.g., Feller (1968)), one concludes that
\[
\varphi_{n,\delta} (t) = e^{-n \delta t} \mu_{t,1} P (t) + \sum_{k=1}^{n-1} \left( \frac{\lambda \mu_k}{c} \right)^k \mu_{t,1} P (t) \varphi_{n,\delta} (t \geq 0).
\]

where \( \mu_{n,m} \) is the probability generating function (pgf) of the pmf \( \{ q_i \}_{i \geq 1} \). We aim at first deriving a closed-form expression for \( f_t (t) \).
Lemma 5. In the classical risk model with claim sizes having LT (20),

\[ g_i(t) = \sum_{n=1}^{\infty} \varepsilon_{t,n} \tau_{\alpha c \beta, \alpha} (t), \]

(21)

where

\[ \varepsilon_{t,n} = \alpha_{t,n} \left( \frac{c \beta}{\lambda + c \beta} \right)^n + \sum_{j=1}^{n} \frac{\beta}{\lambda} \varepsilon_{t,j} \left( \frac{n-1}{j-1} \right) \times \left( \frac{\lambda}{\lambda + c \beta} \right)^{n-j+1} \left( \frac{c \beta}{\lambda + c \beta} \right)^{j-1}, \]

with \( \alpha_{t,n} \) and \( \varepsilon_{t,j} \) defined in (24) and (27) respectively.

Proof. We start with the identification of a closed-form expression for \( f_j \) defined in (14). Using Proposition 1(i), one easily derives that

\[ \mu_{t,i} \mathcal{F}(t) = \sum_{i=1}^{\infty} \eta_{ij} \tau_{\beta, j}(t), \]

(22)

where

\[ \eta_{ij} = \int_{0}^{\infty} y^j k_i(y) dy = \frac{1}{\beta^{j+1}} \sum_{j=0}^{\infty} \frac{q_{ij}(j+1)!}{j!}. \]

Substituting (3) and (22) into (14), one finds that

\[ f_j(t) = \frac{\lambda}{c} \left( \sum_{i=1}^{\infty} \eta_{ij} \tau_{\beta, j}(t) + \sum_{k=1}^{n-1} \frac{\lambda \mu}{c} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \eta_{ij} w_{n-i}^{\infty} \times \int_{0}^{t} \tau_{\beta, j}(w) \tau_{\beta, j}(t-w) dw \right) \]

\[ = \sum_{n=1}^{\infty} \alpha_{t,n} \tau_{\alpha c \beta, \alpha} (t), \]

(23)

where

\[ \alpha_{t,n} = \frac{\lambda}{c} \left( \eta_{n} + \sum_{i=1}^{n-1} \eta_{ij} \sum_{k=0}^{\infty} \left( \frac{\lambda \mu}{c} \right)^k w_{n-i}^{\infty} \right), \]

(24)

and \( w_{n-i}^{\infty} \) is the k-fold convolution of the pmf \( \{ w_j \}_{j \geq 1} \).

We now turn our attention to \( g_i(t) \), and show that it also admits a mixed Erlang form. Indeed, we have

\[ g_i(t) = ce^{-\lambda t} f_i(ct) + \sum_{n=1}^{\infty} \frac{\tau_{\alpha c \beta, \alpha}}{n} \times \int_{0}^{ct} yp^n \left( ct - y \right) f_i(y) dy, \]

(25)

where

\[ \int_{0}^{ct} yp^n \left( ct - y \right) f_i(y) dy \]

\[ = \sum_{i=1}^{\infty} q_i^n \sum_{j=1}^{\infty} \alpha_{t,j} \int_{0}^{ct} y \tau_{\beta, j} \left( ct - y \right) \tau_{\beta, j} (y) dy \]

\[ = \sum_{i=1}^{\infty} q_i^n \sum_{j=1}^{\infty} \frac{\beta}{\lambda} \alpha_{t,j} \int_{0}^{ct} \tau_{\beta, j} \left( ct - y \right) \tau_{\beta, j+1} (y) dy \]

\[ = \sum_{i=1}^{\infty} q_i^n \sum_{j=1}^{\infty} \frac{\beta}{\lambda} \alpha_{t,j-1} \tau_{\beta, j} (ct) \]

\[ = \sum_{j=1}^{\infty} \varepsilon_{t,j} \tau_{\beta, j} (ct), \]

and

\[ \varepsilon_{t,j} = \frac{1}{\beta} \sum_{i=1}^{j-2} q_i^n \alpha_{t,j-i-1} \]

(27)

From (26) and (23), (25) becomes

\[ g_i(t) = \sum_{n=1}^{\infty} \alpha_{t,n} e^{-\lambda t} \tau_{\beta, n} (ct) + \sum_{n=1}^{\infty} \frac{1}{\beta} \sum_{j=1}^{\infty} \varepsilon_{t,j} \tau_{\alpha c \beta, \alpha} (t) \tau_{\beta, j} (ct) \]

\[ = \sum_{n=1}^{\infty} \alpha_{t,n} \left( \frac{c \beta}{\lambda + c \beta} \right)^n \tau_{\alpha c \beta, \alpha} (t) \]

\[ + \sum_{n=1}^{\infty} \frac{1}{\beta} \sum_{j=1}^{\infty} \varepsilon_{t,j} \left( \frac{\lambda \beta c^{j-1}}{(n-1)! (j-1)!} \right) \tau_{\alpha c \beta, \alpha} (t) \tau_{\beta, j} (ct) \]

\[ = \sum_{n=1}^{\infty} \alpha_{t,n} \left( \frac{c \beta}{\lambda + c \beta} \right)^n \tau_{\alpha c \beta, \alpha} (t) \]

\[ + \sum_{n=1}^{\infty} \frac{\beta}{\lambda} \sum_{j=1}^{\infty} \varepsilon_{t,j} \left( \frac{\lambda}{\lambda + c \beta} \right)^{n-j+1} \times \left( \frac{c \beta}{\lambda + c \beta} \right)^{j-1} \tau_{\alpha c \beta, \alpha} (t) \tau_{\beta, j} (ct) \]

Interchanging the order of summation yields (21). \( \Box \)

In the next proposition, we apply an inductive argument to identify the functional form of \( \varphi_{n, \delta} (t) \).

Proposition 6. \( \varphi_{n, \delta} \) has a mixed Erlang representation of the form

\[ \varphi_{n, \delta} (t) = \sum_{j=1}^{\infty} \chi_{n,j} \tau_{\alpha c \beta + m, j} (t), \]

(28)

where the weights \( \chi_{n,j} \) are obtained recursively via

\[ \chi_{n,j} = \varepsilon_{t,j} \left( \frac{\lambda + c \beta}{\lambda + c \beta + k \delta} \right)^j \]

\[ + \sum_{i=1}^{j-1} \left( \frac{j}{i} \right) \sum_{l=1}^{n-i} \varepsilon_{l,j-i} \left( \frac{\lambda + c \beta}{\lambda + c \beta + j \delta} \right)^{n-i} \]

\[ \times \sum_{k=1}^{l} \chi_{l,k} \varepsilon_{k,j} \left( \frac{\lambda + c \beta + l \delta}{\lambda + c \beta + j \delta} \right), \]

with \( \chi_{1,j} = \varepsilon_{1,j} \left( \frac{\lambda + c \beta + j \delta}{\lambda + c \beta + j \delta} \right)^j \).

Proof. We shall first prove (28) at \( n = 1 \). Using (21), it is clear that

\[ \varphi_{1, \delta} (t) = \sum_{j=1}^{\infty} \chi_{1,j} \tau_{\alpha c \beta + k, j} (t) \]

\[ = \chi_{1,1} \tau_{\alpha c \beta + k, 1} (t), \]

where \( \chi_{1,j} = \varepsilon_{1,j} \left( \frac{\lambda + c \beta + j \delta}{\lambda + c \beta + j \delta} \right)^j \). Through an inductive argument, we assume that (28) holds for \( n = 1, 2, \ldots, k-1 \), and subsequently prove that (28) is valid at \( n = k \). Substituting (21) and (28) into (19) at \( n = k \) yields
\[ \psi_{\alpha, \beta}(t) = \sum_{n=1}^{\infty} \sum_{i=0}^{n} e^{-\delta i} \tau_{\alpha + c \beta + n \delta} \left( (1 - e^{-\delta})^{n-1} \right) \left( \frac{\alpha}{\beta} \right)^{n}, \]

for \( t \geq 0 \) and \( n = 1, 2, \ldots \).

### 3.3. Sparre Andersen risk model with exponential claim sizes

When claim sizes are exponentially distributed with mean \( 1/\beta \), the time of the first ascending ladder height and its size (also exponential thanks to the lack-of-memory property) are mutually independent. As a result,

\[ E \left[ e^{-\beta y} \left( L_t \right); \gamma_1 < \infty \right] = \frac{n^1}{\beta n^n} \left[ e^{-\beta y}; \gamma_1 < \infty \right]. \]

Letting \( \phi_\delta = E \left[ e^{-\beta y}; \gamma_1 < \infty \right] \), the substitution of (31) into (9) yields

\[ \tilde{m}_{\delta, 1}(z) = \frac{\phi_{z+n\delta}}{1 - \phi_{z+n\delta}} \sum_{m=0}^{n-1} \frac{n!}{m!} \tilde{m}_{\delta, m}(z). \]

By rearrangements, we obtain

\[ \tilde{m}_{\delta, 1}(z) = \frac{1}{z} \frac{\phi_{z+n\delta}}{1 - \phi_{z+n\delta}}, \]

and

\[ \tilde{m}_{\delta, n}(z) = \frac{\phi_{z+n\delta}}{1 - \phi_{z+n\delta}} \left( \frac{n^1}{\beta} \sum_{m=0}^{n-1} \frac{(n-1)!}{m!} \tilde{m}_{\delta, m}(z) \right) + \frac{n}{\beta} \tilde{m}_{\delta, n-1}(z). \]

for \( n = 2, 3, \ldots \). It follows that

\[ \tilde{m}_{\delta, n}(z) = \frac{1}{z} \frac{n^1}{\beta} \tilde{m}_{\delta, n-1}(z) + \frac{n}{\beta} \tilde{m}_{\delta, n-1}(z), \]

where

\[ C_{n, \delta}(z) = \frac{n}{1 - \phi_{z+n\delta}}, \]

To invert (33) wrt \( z \), we first devote our attention to \( C_{n, \delta}(z) \). By definition,

\[ C_{n, \delta}(z) = \frac{1}{1 - \phi_{z+n\delta}} \tilde{m}_{\delta, n-1}(z), \]

for \( n = 2, 3, \ldots \) where

\[ C_{1, \delta}(z) = \frac{1}{1 - \phi_{z+n\delta}}. \]

From Landriault et al. (2011), \( \phi_\delta \) is the unique solution (in \( x \)) within the unit circle of

\[ x - \tilde{k} (\delta + c \beta (1 - x)) = 0. \]

Using Lagrange's expansion theorem with \( f(x) = (1 - x)^{-1} \) (see Cohen (1969, pp. 624–625)), we obtain

\[ (1 - \phi_\delta)^{-1} = 1 + \int_0^\infty h_\delta(y) \, dy, \]

where

\[ h_\delta(y) = e^{-c\beta y}, \]

and

\[ b(y) = \sum_{n=1}^{\infty} k^n (y) \sum_{j=0}^{n} \frac{n-j+1}{n} (c\beta y)^{j-1} e^{-c\beta y} \]

\[ = \frac{1}{\beta} \sum_{n=1}^{\infty} k^n (y) \sum_{j=0}^{n} \frac{n-j+1}{n} \tau_{\beta, j}(y). \]

Thus

\[ (1 - \phi_{z+j\delta})^{-1} = 1 + \int_0^\infty e^{-c\beta y} h_\delta(y) \, dy. \]

Letting

\[ C_{n, \delta}(z) = 1 + \int_0^\infty e^{-c\beta y} \left( r_{n, \delta}(y) \right) \, dy, \]
it follows that
\[ r_{s,1}(y) = h_s(y), \quad (35) \]
and
\[ r_{s,n}(y) = r_{s,n-1}(y) + h_{ns}(y) + \int_0^y h_{ns}(x) r_{s,n-1}(y-x) \, dx, \quad (36) \]
for \( n = 2, 3, \ldots \). One concludes that
\[
\tilde{m}_{s,n}(x) = \frac{1}{x^n} \left( \int_0^\infty e^{-y} h_{ns}(y) \, dy \right) \times \left( 1 + \int_0^y e^{-y} r_{s,n-1}(y) \, dy \right) \\
= \frac{1}{x^n} \int_0^\infty e^{-y} \left( h_{ns}(y) + \int_0^y h_{ns}(x) r_{s,n-1}(y-x) \, dx \right) \, dy \\
= \frac{1}{x^n} \int_0^\infty e^{-y} \left( r_{s,n}(y) - r_{s,n-1}(y) \right) \, dy. \quad (37)
\]

Inverting (32) and (37) wrt \( z \), we get
\[
m_{s,n}(t) = \frac{n!}{t^n} \left( R_{s,n}(t) - R_{s,n-1}(t) \right),
\]
for \( n = 1, 2, \ldots \), where \( R_{s,0}(t) = 0 \) for all \( t \geq 0 \) and \( R_{s,n}(t) = \int_0^t r_{s,n}(x) \, dx \).

### 3.3.1. Mixed Erlang Interclaim Times

In this subsection, we assume that the interclaim times have a mixed Erlang distribution with LT \( K(s) = C \left( \frac{s}{\lambda + c \beta} \right) \) where \( C(z) = \sum_{i=1}^\infty c_i z^i \) and \( |c_i| \geq 1 \) is a pmf. Let \( c^* \) be the pmf of the pmf \( C \). To obtain an expression for the moments of \( Z_s(t) \), we first examine the function \( b(y) \) defined in (34):
\[
b(y) = \frac{1}{c^*} \sum_{i=1}^\infty \sum_{n=0}^\infty c_i^{*n} r_{s,i}(y) \left( \frac{n-j+1}{n} \right) t_{s,i}(y) \\
= \sum_{j=1}^\infty \sum_{n=0}^\infty c_i^{*n} \left( \frac{n-j+1}{n} \right) t_{s,i}(y) \\
\times \frac{\lambda^j}{(j-1)!} \left( c^* \right)^{j-1} \\
= \sum_{j=1}^\infty \sum_{n=0}^\infty c_i^{*n} \left( \frac{n-j+1}{n} \right) t_{s,i}(y) \\
\times \frac{\lambda^j}{(j-1)!} \left( c^* \right)^{j-1} \\
\times \left( \frac{\lambda + c \beta}{\lambda + c \beta} \right)^{j-1} t_{s,i}(y) \quad (38)
\]
Simple manipulations of (38) result in
\[
b(y) = \sum_{i=0}^\infty \theta_i \tau_{s,i+c^*,i}(y),
\]
where
\[
\theta_i = \sum_{j=0}^{i-1} \left( \frac{\lambda}{\lambda + c \beta} \right)^{i-j} \left( c^* \right)^{j} \sum_{n=i+1}^\infty c_i^{*n} \left( \frac{n-j+1}{n} \right) t_{s,i}(y).
\]
One easily obtains that \( h_{0,s} \) is also of a mixed Erlang form, i.e.
\[
h_{0,s}(y) = \sum_{i=1}^\infty \theta_i \tau_{s,i+c^*,i}(y), \quad (39)
\]
where \( \theta_{i,s} = \theta_i \left( \frac{\lambda + c \beta}{\lambda + c \beta + n} \right)^i. \)

Using (36) together with (39), we propose to first identify the functional form of \( r_{s,j}. \)

**Proposition 8.** \( r_{s,j} \) has a mixed Erlang representation of the form
\[
r_{s,j}(y) = \sum_{i=1}^\infty \pi_{i,j} \tau_{s,i+c^*,i}(y), \quad (40)
\]
where the weights \( \pi_{i,j} \) are obtained recursively via
\[
\pi_{1,j} = \sum_{k=1}^{j-1} \pi_{1-k} c_{i,k} \left( \frac{\lambda + c \beta + (n-1) \delta}{\lambda + c \beta + n \delta} \right) \eta_{i+j,n} \phi_{i+j,n}, \quad (41)
\]
for \( l = 2, 3, \ldots \) where \( \pi_{1,l} = \theta_{l,s}, \).

**Proof.** Using the identity (35) and (39) implies that (40) holds at \( l = 1 \) where \( \pi_{1,1} = \theta_{1,s}. \) Henceforth, we assume that (40) is valid for \( l = 1, 2, \ldots, n-1 \), and show that (40) holds for \( l = n. \)

From the recursion (36), we have that the two RHS functions, namely \( r_{s,n-1} \) and \( h_{ns} \), are respectively expressed in terms of the Erlang densities \( \tau_{s,c^*,+n-1,i} \) and \( \tau_{s,c^*,+n,i}. \) Using Proposition 1, \( r_{s,n-1} \) can be rewritten as
\[
r_{s,n-1}(y) = \sum_{i=1}^\infty \sum_{j=0}^\infty \pi_{i,j} \eta_{i+j,n} \phi_{i+j,n} \left( \frac{\lambda + c \beta + (n-1) \delta}{\lambda + c \beta + n \delta} \right) \tau_{s,i+c^*,i+j-1}(y), \quad (42)
\]
for \( n = 2, 3, \ldots \)

Finally, substituting (39) and (41) into (36) yields
\[
r_{s,n}(y) = \sum_{i=1}^\infty \pi_{n,i} \tau_{s,i+c^*,i}(y), \quad (43)
\]
where
\[
\pi_{n,i} = \eta_{n-1,i} + \theta_{i,s} + \sum_{k=1}^{\infty} \eta_{n-1,i} \phi_{k-j,n} \phi_{i+j,n}. \]

Substituting (42) into (43) completes the proof of this proposition.

**Using (40) at \( l = n \) together with (41), a closed-form expression for the \( n \)-th moment of \( Z_s(t) \) is obtained.

**Proposition 9.** The \( n \)-th moment of \( Z_s(t) \) is given by
\[
m_{s,n}(t) = \frac{n!}{t^n} \sum_{i=1}^\infty \tau_{s,i}(y) - \eta_{n-i,n} \phi_{i+j,n} \left( \frac{\lambda + c \beta + (n-1) \delta}{\lambda + c \beta + n \delta} \right) \tau_{s,i+c^*,i+j-1}(y), \quad (44)
\]
for \( t \geq 0 \) and \( n = 1, 2, \ldots \) where \( \eta_{0,i} = 0(i = 1, 2, \ldots). \)
### 4. Application—distribution of $Z_δ(t)$

Capitalizing on the results developed in the earlier sections, we further analyze the distribution of $Z_δ(t)$ through an approximation technique together with a simulation study. Note that the cdf of $Z_δ(t)$ is of the form

$$F_{Z_δ(t)}(x) = 1 - \psi(0,t) + \psi(0,t)F_{Y_δ(t)}(x),$$

where $\psi(0,t)$ is the probability of ruin within $[0,t]$ with an initial surplus of 0, and $F_{Y_δ(t)}$ is the cdf of a strictly positive rv $Y_δ(t)$ whose moments are given by

$$E[Y_δ(t)^n] = \frac{E[Z_δ(t)^n]}{\psi(0,t)}, \quad (44)$$

for $n = 1, 2, \ldots$. Our objective is to provide a moment-matching method to approximate the distribution of $Y_δ(t)$ given that its moments can be obtained from (44) and the material presented in the earlier sections. Moment-matching techniques have been extensively studied in the literature. The reader is referred to Johnson and Taffe (1989) and Feldmann and Whitt (1998), and references therein.

In this section, we intend to use a technique proposed by Cossette et al. (in preparation) to approximate the distribution of $Y_δ(t)$ which capitalizes on the denseness of the mixed Erlang class (see Tijms (1994)). Indeed, for a given number of moments (say $\nu$), we consider all mixed Erlang densities of the form

$$E[Y_{\beta,M}(t)],$$

where $M$ is a rv with $\nu$ atoms on the set of positive integers $[1, 2, \ldots, \nu]$ for a previously chosen $\nu$ which matches the first $\nu$ moments of $Y_δ(t)$. Note that $\nu$ should be chosen large enough to ensure that a solution exists; see Cossette et al. (in preparation) for further details. Among this set of eligible approximations, as a rule of thumb we select the one that minimizes the difference between their $(\nu+1)$-th moments. As an illustration, we consider the following numerical example.

**Example 10.** Assume that both interclaim times and claim sizes are exponentially distributed with mean 1. We consider a constant premium rate of 1.2 and a force of interest of 4%. The values of the first 6 moments of $Z_δ(t)$ are provided for $t = 10, 50, 100$ and 200 in the following table:

<table>
<thead>
<tr>
<th>n/t</th>
<th>10</th>
<th>50</th>
<th>100</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.0904</td>
<td>2.7911</td>
<td>2.8291</td>
<td>2.8309</td>
</tr>
<tr>
<td>3</td>
<td>71.3258</td>
<td>112.4099</td>
<td>135.4742</td>
<td>135.5999</td>
</tr>
<tr>
<td>4</td>
<td>612.7045</td>
<td>1315.6872</td>
<td>1538.1479</td>
<td>1538.2994</td>
</tr>
<tr>
<td>5</td>
<td>6254.4329</td>
<td>15381.4273</td>
<td>15837.2994</td>
<td>15855.6367</td>
</tr>
<tr>
<td>6</td>
<td>73472.8075</td>
<td>205368.4629</td>
<td>212098.8478</td>
<td>212364.9329</td>
</tr>
</tbody>
</table>

As expected, we observe that the moments of $Z_δ(t)$ converge as $t$ becomes large, since $Z_δ(t)$ converges to $Z_∞$ due to the presence of the strictly positive solvency margin. In the following table, we also give the values of the probability that no ascending ladder heights occurs in $[0,t]$ for $t = 10, 50, 100, 200$ which immediately lead to the values of the moments of $Y_δ(t)$:

<table>
<thead>
<tr>
<th>$t$</th>
<th>10</th>
<th>50</th>
<th>100</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - \psi(0,t)$</td>
<td>0.25519</td>
<td>0.18129</td>
<td>0.16996</td>
<td>0.16738</td>
</tr>
</tbody>
</table>

The values of $\psi(0,t)$ are obtained with 100,000 Monte Carlo simulations of $Z_δ(t)$.

For the approximation, we consider a time horizon of $t = 200$ and assume that $M$ is a rv with atoms on the integers $\{1, 2, \ldots, 10\}$. For convenience, assume that $M$ has pmf $\rho$. Based on the approximation technique described above, the following mixed Erlang models were found to approximate $Y_δ(t)$ when the first $\nu$ moments are matched:

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$\rho$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\rho_3$</th>
<th>$\rho_4$</th>
<th>$\rho_5$</th>
<th>$\rho_6$</th>
<th>$\rho_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.9662</td>
<td>-</td>
<td>0.7428</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.2572</td>
</tr>
<tr>
<td>3</td>
<td>0.7578</td>
<td>0.3829</td>
<td>-</td>
<td>0.5028</td>
<td>-</td>
<td>-</td>
<td>0.1143</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>0.7911</td>
<td>0.3033</td>
<td>0.2899</td>
<td>-</td>
<td>0.3469</td>
<td>-</td>
<td>-</td>
<td>0.0600</td>
</tr>
<tr>
<td>5</td>
<td>0.7452</td>
<td>0.3507</td>
<td>0.1543</td>
<td>0.2816</td>
<td>0.1249</td>
<td>-</td>
<td>0.0884</td>
<td>-</td>
</tr>
</tbody>
</table>

In Fig. 2, we compare the cdf of these 4 mixed Erlang distributions with the empirical cdf that resulted from 10 million Monte Carlo simulations of $Z_δ(t)$.

Clearly, the approximations obtained with 3 and more moments are very good. Also, for these four models, we compare the values of...
Table 1

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\delta$</th>
<th>$\kappa$</th>
<th>$\delta$</th>
<th>$\kappa$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.95</td>
<td>0.99</td>
<td>0.995</td>
<td>0.95</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\delta$</th>
<th>$\kappa$</th>
<th>$\delta$</th>
<th>$\kappa$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.95</td>
<td>0.99</td>
<td>0.995</td>
<td>0.95</td>
<td>0.99</td>
</tr>
<tr>
<td>2</td>
<td>11.1205</td>
<td>14.1240</td>
<td>15.2955</td>
<td>11.1108</td>
<td>14.6846</td>
</tr>
</tbody>
</table>

VaR$_{\delta}$ (Z$_{\delta}$ (200)) and TVaR$_{\kappa}$ (Z$_{\kappa}$ (200)) with those obtained through the simulation study (see Tables 1 and 2).

With the first 5 moments matched, one observes that the VaR and TVaR risk measures compare very well with their simulated counterparts.

Acknowledgments

Support for Helene Cossette, David Landriault and Etienne Marceau from grants from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged. Support from the Chaire d’actuariat de l’Université Laval is also gratefully acknowledged by Hélène Cossette, Etienne Marceau, and Khouzeima Moutanabbir.

References


