Abstract

In this paper, we study the discrete time renewal risk model, an extension to Gerber's compound binomial model. Under the framework of this extension, we study the aggregate claim amount process and both finite-time and infinite-time ruin probabilities. For completeness, we derive an upper bound and an asymptotic expression for the infinite-time ruin probabilities in this risk model. Also, we demonstrate that the proposed extension can be used to approximate the continuous time renewal risk model (also known as the Sparre Andersen risk model) as Gerber's compound binomial model has been proposed as a discrete-time version of the classical compound Poisson risk model. This allows us to derive both numerical upper and lower bounds for the infinite-time ruin probabilities defined in the continuous time risk model from their equivalents under the discrete time renewal risk model. Finally, the numerical algorithm proposed to compute infinite-time ruin probabilities in the discrete time renewal risk model is also applied in some of its extensions.

Keywords: Ruin theory; Renewal risk model; Ruin probabilities; Upper bound; Numerical approximation; Variable premiums

1. Introduction

In this paper, we consider the discrete time renewal risk model, an extension to Gerber's compound binomial model, which has been studied in the actuarial literature by Pavlova and Willmot (2004) and Li (2005) among others. Under the assumptions of the compound binomial model, interclaim times have a geometric distribution. In the discrete time renewal risk model, we rather assume that the claim arrivals are governed by a (general) discrete time renewal process. Clearly, this extension for discrete time risk models is similar to the generalization of the classical compound Poisson risk model to the continuous time renewal risk model also known as the Sparre Anderson risk model (see e.g. Rolski et al., 1999).

In the discrete time renewal risk model, the claim number process \( N = \{ N_k, k \in \mathbb{N}^+ \} \) is a renewal process with interclaim times \( \{ T_j, j \in \mathbb{N}^+ \} \) where \( \{ T_j, j \in \mathbb{N}^+ \} \) is a sequence of independent and strictly positive integer-valued r.v.'s. \( N_k \) is called an ordinary renewal process if the r.v.'s \( \{ T_j, j \in \mathbb{N}^+ \} \) are identically distributed with probability mass function (p.m.f.) \( f_T \) and cumulative distribution function (c.d.f.) \( F_T(x) = 1 - F_T(x) \). Letting the r.v. \( T_0 \) have the same distribution as the r.v.'s \( T_1, T_2, \ldots \) is equivalent to assuming that a renewal has just occurred at time 0. If instead we
keep the same assumptions but assume that the distribution of $T_1$ differs from that of $T_2, T_3, \ldots$, $N$ is referred to as a delayed renewal process. A common choice for the distribution of $T_1$ is the equilibrium distribution for which $N$ is a stationary renewal process. We have

$$f_1(k) = \frac{F_1(k-1)}{E(T)} \quad k \in \mathbb{N}^+$$

when $N$ is a stationary renewal process.

The individual claim amount r.v.'s $X_j$, $j \in \mathbb{N}^+$, where $X_j$ corresponds to the amount of the $j$th claim, are assumed to be a sequence of strictly positive, independent and identically distributed (i.i.d.) r.v.’s. $f_X$ and c.d.f. $F_X(x) = 1 - F_X(x)$ Moreover, we assume that the r.v.'s $T_1, T_2, \ldots$ and $X_1, X_2, \ldots$ are mutually independent. The total claim amount process $S = \{S_k, k \in \mathbb{N}\}$ is defined as $S_k = \sum_{j=1}^{N_k} X_j$, where $N_k$ equals $0$ if $k < a$.

Finally, we define the surplus process $U = \{U_i, k \in \mathbb{N}\}$ as $U_0 = u$ and $U_k = u + e_k - S_k$ for $k \in \mathbb{N}^+$. $U_k$ is the level premium received per period. For discrete time risk models, there are two distinct definitions of the time of ruin in the literature. In the first one, it is assumed that ruin occurs when the surplus falls below zero while in the second, ruin is assumed to occur when the surplus falls below or at the level 0 (see e.g. Shiu, 1989 and Gerber, 1988). In this paper, we consider only the first one, namely, $\tau = \inf\{k \geq 1 : U_k < 0\}$ is defined as the time of ruin associated to $U$ with $\tau = \infty$ if $U_k \geq 0$ for all $k \in \mathbb{N}$ (i.e. ruin does not occur). Let $\phi(u, a) = E[1_{\tau \leq a}]$ and $\psi(u) = E[1_{\tau = \infty}]$ be the finite-time and infinite-time ruin probabilities where the indicator function $1_A = 1$ if $A$ is true and 0 otherwise. Their complements, the finite-time and infinite-time non-ruin probabilities, are respectively denoted by $\phi(u, a)$ and $\phi(u)$. To ensure that $\phi(u)$ in the discrete time renewal risk model goes to 0 as $u \to \infty$, the premium rate is such that

$$cE(T) > E[X].$$

The paper is structured as follows: in Sections 2 and 3, we study the aggregate claim amount process and the finite-time and infinite-time ruin probabilities in the discrete time renewal risk model. In Section 4, an upper bound and an asymptotic expression for the infinite-time ruin probabilities in the discrete time renewal risk model are provided for completeness. In Section 5, we derive both numerical upper and lower bounds for the infinite-time ruin probabilities in the continuous time renewal risk model through the infinite-time ruin probabilities in the discrete time renewal risk model. Finally, in Section 6, we briefly present some additional results under the framework of the discrete time renewal risk model and in some of its extensions.

2. Aggregate claim amount process

In this section, we study the aggregate claim amount process in the discrete time renewal risk model. In this paper, most results will be derived in the ordinary case since the delayed case can easily be deduced from the ordinary one by noting that the claim number process just beginning at time $T_1$ is an ordinary renewal process.

When $N$ is an ordinary renewal process, it is well known that $E[N_k] = E[X]E[N]$ where $E[N] = \sum_{j=1}^{\infty} F_X(k)$ is the solution to the discrete time renewal equation

$$E[N_k] = F_T(k) + \sum_{j=1}^{k} E[N_{k-j}]f_T(j).$$

for $k \in \mathbb{N}^+$ (see Rolski et al., 1999) assuming $E[N_0] = 0$. When $N$ is a stationary renewal process, we have $E[N_k] = E[X]E[N]$ where, in this case, $E[N_k] = \frac{k}{E(T)}$ is obtained by induction. Clearly, $E[N_0] = f_T(1) = 1/E(T)$. Then, suppose that $E[N_{j+k}] = \frac{j}{E(T)}$ holds for $j \in [2, \ldots, k - 1]$. By first conditioning on the r.v. $T_1$ with p.m.f. (1), one obtains

$$E[N_{j+k}] = \sum_{i=1}^{k} f_T(i) + \sum_{i=1}^{j} f_T(i)E[N_{k-i}].$$

Therefore, $E[N_{j+k}] = \frac{j}{E(T)}$. Consequently, the infinite-time renewal equation for a stationary renewal process $N$ is

$$E[N_k] = \frac{1}{E(T)} + \sum_{j=1}^{\infty} \frac{1}{E(T)} f_T(j) = \frac{1}{E(T)}$$. 

\[E[N_k] = E[T] = \sum_{j=1}^{\infty} j f_T(j) = \frac{1}{E(T)}\]

The above result has been pointed out by many authors such as Eeckhoudt and Pavlova (1993) and Eeckhoudt et al. (1995b).
Replacing (3) in (4), we have
\[
E[N^k_k] = \sum_{i=1}^{k} f^k_i(i) + \sum_{j=1}^{k} f^k_j(j) \left\{ \sum_{i=1}^{k-i} f^k_{i} + \sum_{i=1}^{k-1} E[N_{k-i-1}]f^k_i(j) \right\}
\]
and reverting the order of summation, we obtain
\[
E[N^k_k] = \sum_{i=1}^{k} f^k_i(i) + \sum_{j=1}^{k} f^k_j(j) \left\{ \sum_{i=1}^{k-i} f^k_{i} + \sum_{i=1}^{k} E[N_{k-j-1}]f^k_i(i) \right\}
\]
\[
= \sum_{i=1}^{k} f^k_i(i) + \sum_{i=1}^{k} E[N^k_0]f^k_i(i)
\]
\[
= \sum_{i=1}^{k} f^k_i(i) + \sum_{i=1}^{k} E[N^k_0]f^k_i(i)
\]
Using an inductive argument, it follows that
\[
E[N^k_k] = \sum_{i=1}^{k} \frac{F^k(i-1)}{E[T]} + \sum_{i=1}^{k} \frac{k-i}{E[T]} \cdot f^k_i(i) = \frac{k}{E[T]}
\]
In the next proposition, a recursive algorithm is provided to compute the p.m.f. of \( S_k \).

**Proposition 1.** In the discrete time renewal risk model with \( N \) an ordinary renewal process, the p.m.f. of \( S_k \), \( f^k_k \), for \( k \in \mathbb{N}^+ \) is given by
\[
f^k_k(m) = \begin{cases} F^k(k), & m = 0 \\ \sum_{j=1}^{m} f^k(j) \sum_{i=1}^{m} f^k(i) f^k_{i-j}(m-i), & m \in \mathbb{N}^+ \end{cases} \quad (5)
\]
where \( f^k_0(t) = 1_{t<0} \).

**Proof.** We condition on the r.v.’s \( T_j \) and \( X_j \) and the result follows from the stationarity of the aggregate claim amount process at renewal epochs.

\[ \square \]

### 3. Ruin and non-ruin probabilities

The main aim of this paper is to propose ways to compute the finite-time and infinite-time (non-) ruin probabilities in the discrete time renewal risk model. Therefore, we propose next a recursive algorithm to compute the finite-time non-ruin probabilities and a numerical algorithm to obtain the infinite-time non-ruin probabilities.

**Proposition 2.** In the discrete time renewal risk model with \( N \) an ordinary renewal process, the finite-time non-ruin probabilities can be computed with
\[
\phi(u, n) = \sum_{j=1}^{n} f^u(j) \sum_{i=1}^{n} f^u(i) \phi(u + i, n - j) + \Phi^u(n),
\]
for \( u \in \mathbb{N}, n \in \mathbb{N}^+ \) and with \( \phi(x, 0) = 1_{x \in \mathbb{N}} \).

**Proof.** We condition on both r.v.’s \( T_j \) and \( X_j \) and from the stationarity of the surplus process at renewal epochs, the result follows.

\[ \square \]

Similar to Cossette et al. (2004), we propose a numerical algorithm to obtain the infinite-time ruin probabilities in the discrete time renewal risk model. This algorithm can be viewed as a general numerical method to approximate
infinite-time ruin probabilities and can be applied in most discrete time risk models with the property \( \phi(u) \to 1 \) as \( u \to \infty \). The main idea of the numerical algorithm will be to let the infinite-time non-ruin probabilities equal 1 when the initial surplus is greater than \( u_0 \) for a value of \( u_0 \in \mathbb{N} \), which is preferably chosen large.

We give in what follows the main steps that lead to our proposed approximation. By conditioning on both r.v.’s \( T_1 \) and \( X_1 \) and with the stationarity of the surplus process at renewal epochs, we know

\[
\phi(u) = \sum_{j=1}^{\infty} f_X(j) \sum_{k=0}^{\infty} \phi(k) f_X(u + cj - k). \tag{6}
\]

- **Step 1**: From (2), we know that \( \lim_{u \to \infty} \phi(u) = 1 \) in the discrete time renewal risk model for \( u \in \mathbb{N} \). We therefore approximate \( \phi(u), u \in \mathbb{N} \) in (6) by letting the non-ruin probabilities be equal to 1 for \( u \geq u_0 \) where \( u_0 \) is chosen large.

- **Step 2**: Given the infinite sum on \( j \) in (6), we also choose a (large) value, say \( t_0(\bar{t}_0 \in \mathbb{N}) \), for which if no claim occurs during the next \( t_0 \) periods, we assume that ruin can a.s. no longer occur.

Combining these two steps to (6), we propose to approximate \( \phi(u), u \in \mathbb{N} \) by \( \phi^*(u), u \in \mathbb{N} \) where

\[
\phi^*(u) = 1, \tag{7}
\]

for \( u = u_0 \) and where \( \phi^*(u) \) for \( u = 0, \ldots, u_0 - 1 \) is obtained by solving the system of \( u_0 \) equations-\( u_0 \) unknown parameters built from the equality

\[
\phi^*(u) = \sum_{j=1}^{u_0} f_X(j) \sum_{k=0}^{u_0-1} \phi^*(k) f_X(u + cj - k) + \sum_{j=1}^{u_0} f_X(j) \sum_{k=u_0}^{u_0-1} f_X(u + cj - k) + \bar{F}_t(t_0), \tag{8}
\]

for \( u = 0, \ldots, u_0 - 1 \) where \( m \times n = \min(m, n) \).

In the following proposition, we quantify the error made in the approximation of \( \phi(u), u \in \mathbb{N} \) by \( \phi^*(u), u \in \mathbb{N} \).

We rely on \( \sup_{u \in \mathbb{N}} |\hat{\psi}(u)| \) with \( \hat{\psi}(u) = \phi^*(u) - \phi(u) \) for \( u \in \mathbb{N} \) to quantify the largest difference between any exact infinite-time non-ruin probability and its proposed approximation. We consider separately the cases for which the distribution of \( T \) has a finite or an infinite support. For convenience, let \( \psi^*(u) \) be an arbitrary upper bound to the infinite-time ruin probability \( \psi(u) \).

**Proposition 3.** In the discrete time renewal risk model with \( \mathbb{N} \) an ordinary renewal process and \( T \) a finite r.v., we have

\[
\sup_{u \in \mathbb{N}} |\hat{\psi}(u)| \leq \delta,
\]

where \( u_0 = \inf_{u \in \mathbb{N}} \{u : \psi^*(u) \leq \delta\} \) and \( t_0 = \inf_{t \in \mathbb{N}} \{t : \bar{F}_t(t) = 0\} \) in \( \phi^*(u), u \in \mathbb{N} \).

**Proof.** First, from (7), we have

\[
\sup_{u \in \mathbb{N}} |\hat{\psi}(u)| = \sup_{u_0 \in \mathbb{N}, u_0 + 1, \ldots} \psi(u) = \psi(u_0), \tag{10}
\]

We therefore need an expression for \( \Delta = \sup_{u \in \mathbb{N}, u_0 + 1, \ldots} |\hat{\psi}(u)| \). From (6), one knows

\[
\phi(u) = \sum_{j=1}^{u_0} f_X(j) \sum_{k=0}^{u_0-1} \phi(k) f_X(u + cj - k) + \sum_{j=1}^{u_0} f_X(j) \sum_{k=u_0}^{u_0-1} \phi(k) f_X(u + cj - k) + \sum_{j=1}^{\infty} f_X(j) \sum_{k=0}^{\infty} \phi(k) f_X(u + cj - k). \tag{11}
\]
for $u \in \mathbb{N}$. Subtracting (11) from (8), one finds

\[
\xi(u) = \sum_{j=1}^{n} f_T(j) \sum_{k=0}^{u+j(u_0)-1} \psi(k) f_X(u + c - k) + \sum_{j=1}^{n} f_T(j) \sum_{k=u+j(u_0)}^{u+j-1} \psi(k) f_X(u + c - k)
\]

\[
+ \sum_{j=1}^{n} f_T(j) \left( \sum_{u+j(u_0)}^{u+j-1} \psi(k) f_X(u + c - k) \right).
\]

(12)

for $u = 0, \ldots, u_0 - 1$. Then, we take the absolute value on each side of (12), noting that $\xi(k)$ is the only term on the right-hand side of (12) that can potentially be negative. It follows that

\[
|\xi(u)| \leq \sum_{j=1}^{n} f_T(j) \sum_{k=0}^{u+j(u_0)-1} |\xi(k)| f_X(u + c - k) + \sum_{j=1}^{n} f_T(j) \sum_{k=u+j(u_0)}^{u+j-1} \psi(k) f_X(u + c - k)
\]

\[
+ \sum_{j=1}^{n} f_T(j) \left( \sum_{u+j(u_0)}^{u+j-1} \psi(k) f_X(u + c - k) \right).
\]

(13)

By replacing $\psi(u)$ by $\psi(u_0)$ and $\psi(k)$ by 1 in, respectively, the second and third terms on the right-hand side of (13), one derives a larger upper bound for $|\xi(u)|$

\[
|\xi(u)| \leq \sum_{j=1}^{n} f_T(j) \sum_{k=0}^{u+j(u_0)-1} |\xi(k)| f_X(u + c - k) + \psi(u_0) \sum_{j=1}^{n} f_T(j) f_X(u + c - u_0) + F_T(t_0),
\]

(14)

for $u = 0, \ldots, u_0 - 1$. Note that, to this point, no assumption on the r.v. $T$ has been formulated.

When $T$ is a finite r.v., choose $n_0 = \inf \{ t \in \mathbb{N} : F_T(t) = 0 \}$. Suppose that

\[
\Delta > \psi(u_0),
\]

(15)

and we prove by a contradiction argument that it cannot be true. Taking (14) at $u = u_0 - 1$ and then using (15), one deduces

\[
|\xi(u_0 - 1)| \leq \sum_{j=1}^{n_0} f_T(j) \sum_{k=0}^{u_0} \Delta f_X(u_0 + c - (k + 1)) + \psi(u_0) \sum_{j=1}^{n_0} f_T(j) f_X(c - 1)
\]

\[
< \sum_{j=1}^{n_0} f_T(j) \sum_{k=0}^{u_0} \Delta f_X(u_0 + c - (k + 1)) + \Delta \sum_{j=1}^{n_0} f_T(j) f_X(c - 1).
\]

(16)

The strict inequality in (16) is the consequence of (15) and the fact that

\[
\sum_{j=1}^{n_0} f_T(j) f_X(c - 1) > 0.
\]

(17)

Indeed, (17) needs to be true, otherwise the surplus level will a.s. decrease between two consecutive claims. An equivalent statement would be to say that the claim amount will a.s. be greater than or equal to $c$ times the interarrival time between two claims which cannot be true under (2). We therefore conclude, from (16), that

\[
|\xi(u_0 - 1)| < \Delta.
\]

To prove that $|\xi(k)| < \Delta$ for all $k \in \{0, \ldots, u_0 - 1\}$ under (15), let us rely on an inductive argument. We assume that

\[
|\xi(u_0 - j)| < \Delta.
\]

(18)
for \(j = 1, \ldots, l - 1\) and we want to prove that \(|\xi(u_0 - l)| < \Delta\) for \(l = 1, \ldots, u_0\). Replacing \(u\) by \(u_0 - l\) in (14), one finds

\[
|\xi(u_0 - l)| \leq \sum_{j=1}^{u_0} \sum_{k=0}^{u_0-l} \xi(k) \cdot f_X(u_0 + cj - (k + l)) + \psi(u_0) \sum_{j=1}^{u_0} \xi(k) \cdot f_X(cj - l)
\]

and combining this with (10), one deduces

\[
|\xi(u_0 - l)| \leq \sum_{j=1}^{u_0} \sum_{k=0}^{u_0-l} \Delta f_X(u_0 + cj - (k + l)) + \Delta \sum_{j=1}^{u_0} f_X(cj - l) - 1,
\]

from which it clearly follows that \(|\xi(u_0 - l)| < \Delta\). Therefore, \(|\xi(k)| < \Delta\) for all \(k \in \{0, \ldots, u_0 - 1\}\), which cannot be true from the definition of \(\Delta\). We must therefore have \(\Delta \leq \psi(u_0)\) and, when combined to (10), this implies that

\[
\sup_{a \in \mathbb{N}} |\xi(a)| \leq \psi(u_0).
\]

From the definition of \(u_0\) in the statement of the proposition, \((9)\) directly follows. \(\square\)

In Proposition 4, we consider the case where \(T\) is a r.v. with an infinite support. But first, we give some definitions that are required to state the error bound. Let \(b_{x,y}(u) = \sum_{j=1}^{u} f_T(j) F_X(u + cj - y)\), \(d_I(u) = \sum_{j=1}^{u} f_T(j) F_X(u + cj)\) and, finally,

\[
e_{x,y}(u) = \sup_{x \in \{0, \ldots, y-1\}} \frac{F_T(x)}{b_{x,y}(u) + d_I(u) + F_T(x)},
\]

for \(x, y, u \in \mathbb{N}\).

**Proposition 4.** In the discrete time renewal risk model with \(\mathcal{N}\) an ordinary renewal process and \(T\) a r.v. with an infinite support,

\[
\sup_{a \in \mathbb{N}} |\xi(a)| \leq \delta + \epsilon,
\]

where \(u_0 = \inf_{a \in \mathbb{N}} \{u : \psi(u) \leq \delta\}\) and \(t_0 = \inf_{a \in \mathbb{N}} \{t : \epsilon a(t) \leq \epsilon\}\) in \(\Phi^a(u), u \in \mathbb{N}\).

**Proof.** Let \(k_0 \in \{0, \ldots, u_0 - 1\}\) be such that \(\Delta = |\xi(k_0)|\). From (14), one deduces

\[
\Delta \leq \Delta_{a_{x,y}(u_0)(k_0)} + \psi(u_0) a_{x,y}(u_0) + F_T(t_0),
\]

where \(a_{x,y}(u) = \sum_{j=1}^{u} f_T(j) \sum_{s=0}^{l+1} f_X(u + cj - k)\) for \(x, y, u \in \mathbb{N}\). For \(T\) a r.v. with an infinite support, we know that \(a_{x,y}(u) < 1\) for all \(r \in \{0, \ldots, u_0 - 1\}\) and \(a_0, t_0 \in \mathbb{N}\), which ensures that \(a_{x,y}(u_0) < 1\). Therefore, (21)
can be rewritten as
\[
\Delta \leq \frac{\psi(u_0) b_{0,u_0}(k_0) + \bar{F}(t)}{1 - a_{0,u_0}(k_0)} \\
\leq \psi(u_0) + \frac{b_{0,u_0}(k_0) + d_s(k_0)}{1 + \bar{F}(t)}.
\]
However, with no knowledge on the value of \(k_0\), a higher upper bound for \(\Delta\) is given by
\[
\Delta \leq \psi(u_0) + e_u(t).
\] (22)
From the definitions of \(u_0\) and \(t_0\) as stated in the proposition, (22) leads to (20) \(\Box\).

The results contained in both Propositions 3 and 4 are useful to ensure that the approximation \(\phi^\ast\) to \(\phi\) will be sufficiently good. Indeed, by specifying a level of precision (either \(\delta\) or \(\epsilon\) depending on the case), we can choose the values \(u_0\) and \(t_0\) in the approximation \(\phi^\ast\), which guarantees this precision level. It should be noted that \(u_0\) is found in both Propositions 3 and 4 by using an upper bound to the infinite-time ruin probabilities since we do not know the exact value of \(\psi(u_0)\) in both (19) and (22). For the light-tail case, an upper bound is proposed in Section 4.

For any \(\delta > 0\), the existence of \(u_0\) is assured if \(\lim_{u \to \infty} \psi(u) = 0\). In the case where the r.v. \(T\) has an infinite support, \(t_0\) exists for every \(t > 0\) if \(\lim_{u \to \infty} e_u(t) = 0\) which is true since, for a given \(u_0\), it can be proven that there exists a \(r^\ast\) such that \(b_{i,u_0}(t) > 0\) for all \(i \in \{0, \ldots, u_0 - 1\}\) and \(t > r^\ast\). Moreover, since an upper bound for \(\psi(u)\) is usually a decreasing function in \(u\), we conclude that as \(u_0\) increases, \(\delta\) decreases. When the r.v. \(T\) has an infinite support, it can be proven that, for a given \(u_0\), \(e_u(t)\) is a decreasing function in \(t\) which implies that as \(t_0\) increases, \(\epsilon\) decreases.

As we can expect, an increase in the values of \(u_0\) and \(t_0\) leads to a better approximation.

For the delayed case, the infinite-time non-ruin probabilities are obtained by first conditioning on the r.v.'s \(Y_1\) and \(\bar{Y}_k\) and then using the infinite-time non-ruin probabilities in the ordinary case computed via our proposed approximation.

Numerical applications of our proposed approximations will be performed later to evaluate the infinite-time ruin probabilities within the continuous time renewal risk model.

As an alternative to our numerical algorithm, a simulation method based on a change of measure technique, as in the continuous time renewal risk model, could be used to provide an approximation to the infinite-time ruin probabilities in the discrete time renewal risk model (see Rolski et al., 1999).

4. Upper bound and asymptotic expression for ruin probabilities

In this section, we derive an upper bound and an asymptotic expression for \(\psi(u), u \in \mathbb{N}\) in the discrete time renewal risk model when \(\mathbb{N}\) is an ordinary renewal process using an approach similar to Willmot and Lin (2001). More precisely, using the fact that the infinite-time ruin probability is a compound geometric tail in this risk model and results in Willmot (1989) and Willmot and Lin (2001) to achieve our objectives. The results of this section are given for purposes of completeness (due to the error bound of Proposition 3) and the proofs of the upper bound and asymptotic expression for \(\psi(u)\) differ from the ones usually proposed.

Let us denote \(Y_k = X_k - aT_k\) for \(k \in \mathbb{N}\) and \(V_k = \sum_{j=1}^{k} Y_j\) for \(k \in \mathbb{N}\). It is well-known using standard martingale arguments that the sequence \(\{e^{-\mu j - \frac{\mu^2}{2}}; k \in \mathbb{N}\}\) is an exponential martingale with \(\mu\) a real solution to
\[
E[e^{\mu T - \frac{\mu^2}{2}}] = 1.
\] (23)
In the discrete time renewal risk model, there may exist only a unique solution \(r\) different of 0 to (23). This solution, denoted \(\rho\) (if it exists) and called a Lundberg exponent, is strictly greater than 0.

Let us also introduce \(h, H\) and \(R\) as respectively the mass function, cumulative function and generating function of the ladder height distribution. This distribution is a very useful tool when studying random walks (see e.g. Rolski et al., 1999) and corresponds in our risk model to the distribution of the r.v. \(Y_1 + \cdots + Y_{n^\ast}\) where \(n^\ast = \min\{n: Y_1 + \cdots + Y_n > 0\}\) is called an ascending ladder epoch. A result that will be useful later is stated here.

Lemma 5. In the discrete time renewal risk model with \(\mathbb{N}\) an ordinary renewal process, \(\sum_{j=1}^{\infty} e^{\rho j} h(j) = 1\).

Proof. The result is derived from (23) and Lemma 6.4.2. in Rolski et al. (1999). \(\Box\)
Let \( \hat{\psi}(z) = \sum_{u=0}^{\infty} z^u \psi(u) \) be the generating function of the infinite-time ruin probability. Conditioning \( \psi(u) \) on the first ascending ladder epoch \( u^* \), one deduces
\[
\sum_{u=0}^{\infty} z^u \psi(u) = \sum_{u=0}^{\infty} z^u \sum_{j=1}^{\infty} \psi(u - j) h(j) + \sum_{u=0}^{\infty} z^u (H(\infty) - H(u))
\]
\[
= \hat{\psi}(z) \hat{h}(z) + \frac{\hat{\psi}(0) - \hat{h}(z)}{1 - z}
\]
(S24)

Simple modifications to (24) lead to
\[
\hat{\psi}(z) = \frac{1 - \hat{h}(z)}{1 - z}.
\]
where \( \hat{h}(z) = (1 - \hat{\psi}(0))/(1 - \hat{\psi}(0) \hat{g}(z)) \) and \( \hat{g}(z) = \hat{h}(z)/\hat{\psi}(0) \) which implies that \{\( \hat{\psi}(u), u \in \mathbb{N} \)\} is a compound geometric tail. Therefore, to apply results of Chapter 7 in Willmot and Lin (2001), we must first have \( \hat{\psi}(0) = 0 \) (which is verified in our case) and find \( \infty > 1 \) such that \( \hat{g}(z) = (\hat{\psi}(0))^{-1} \) or equivalently that \( \hat{h}(z) = 1 \). From Lemma 5, it is easy to see that \( z = e^\rho \) where \( \rho \) is the Lundberg exponent (if it exists). From the property of \( \rho \), we know that this solution \( z \), different from 1, (if it exists) is unique and strictly greater than 1. In the next two propositions, we derive an upper bound and an asymptotic expression for \( \hat{\psi}(u) \) assuming the existence of \( \infty > 1 \) such that \( \hat{h}(z) = 1 \).

**Proposition 6.** In the discrete time renewal risk model with \( \mathbb{N} \) an ordinary renewal process,
\[
\psi(u) \leq \left( \frac{1}{1 - \infty} \right)^{u+1}.
\]

**Proof.** The result follows from the compound geometric tail of \{\( \hat{\psi}(u), u \in \mathbb{N} \)\} and Corollary 7.2.1 of Willmot and Lin (2001).

This result can also be obtained using inductive arguments.

**Proposition 7.** In the discrete time renewal risk model with \( \mathbb{N} \) an ordinary renewal process, an asymptotic expression for \{\( \hat{\psi}(u), u \in \mathbb{N} \)\} is given by
\[
\hat{\psi}(u) \sim \frac{1 - \hat{\psi}(0)}{\infty + 1} \sum_{j=1}^{\infty} \rho^{j-1} h(j) \left( \frac{1}{\infty} \right)^{u+1}, \quad u \to \infty.
\]

**Proof.** The result follows from the compound geometric tail of \{\( \hat{\psi}(u), u \in \mathbb{N} \)\} and Theorem 2 in Willmot (1989).

**5. Ruin probabilities in the continuous time renewal risk model**

In the continuous time renewal risk model, there exist some cases for which the ruin probabilities have an explicit expression or can be calculated without too much difficulty. An example for the former is when the claim amount r.v.’s and the interclaim times have both a phase-type distribution. In the case where only the claim amount r.v.’s is phase-type, the infinite-time ruin probabilities have a phase-type representation where the vector of initial probabilities is the unique solution of a fixedpoint problem which can be solved numerically (see Asmussen (2000)). In many other cases, there is no explicit expression for the ruin probability. One solution proposed is to resort to a numerical approach such as an efficient Monte Carlo simulation (e.g. Rolski et al. (1999)) which requires, in such a case, the identification of a martingale and a change of measure procedure. Also, with this approach, we have no idea if this estimate understates or overstates the “true” ruin probability. In this section, we propose an alternative to this approach which is based on the discrete time renewal risk model. The attractiveness of this procedure is to lead to both theoretical upper and lower bounds for the infinite-time ruin probabilities in the continuous time renewal risk model. As we will show, those bounds can be improved as desired. Even if we have recourse to a numerical method, we use results of Proposition 3 to obtain genuine numerical bounds for the infinite-time ruin probabilities in the continuous-time risk model. Finally, as we will see, our approach has the advantage that it can be applied with any interclaim time and individual claim amount distributions.
5.1. Continuous time renewal risk model

We briefly introduce the framework underlying the continuous time renewal risk model in which we add the superscript $c$ to quantities or r.v.’s to distinguish them from the same ones in the discrete time renewal risk model. The continuous time renewal risk model is similarly defined as the discrete time renewal risk model except that the interclaim times and the independent claim amounts are no longer assumed to be integer-valued r.v.’s. The claim number process $S^c = \{N^c(t), t \geq 0\}$ is an ordinary renewal process with i.i.d. interarrival times $\{T^c_j, j \in \mathbb{N}^+\}$ and $\{X^c_j, j \in \mathbb{N}^+\}$ is assumed to be a sequence of positive and i.i.d. claim amount r.v.’s distributed like a generic r.v. $X^c$. The r.v.’s $T^c_1, T^c_2, \ldots$ and $X^c_1, X^c_2, \ldots$ are supposed mutually independent. The total claim amount process $S^c = \{S^c(t), t \geq 0\}$ is defined as $S^c(t) = \sum_{j=1}^{N^c(t)} X^c_j$.

We define the surplus process $U^c = \{U^c(t), t \geq 0\}$ with $U^c(t) = u + c^\ast t - S^c(t)$ where the initial surplus level is $u$ with a level premium rate $c^\ast \in \mathbb{R}^+$. The time of ruin $t^\ast$ related to the surplus process $U^c$ is defined as $t^\ast = \inf_{t \geq 0} \{U^c(t) < 0\}$. The infinite-time ruin probability is then defined as

$$
\psi^*(u) = \Pr \left( \max_{u \leq a \leq t} \left\{ \sum_{i=1}^{N^c(t)} (X^c_i - c^\ast T^c_i) \right\} > u \right).
$$

(26)

To ensure that $\psi^*(u) \to 0$ as $u \to \infty$, the premium rate $c^\ast$ must satisfy $c^\ast E[T^\ast] > E[X^c]$.

5.2. Approximation of $\psi^*$

In the present section, we propose an algorithm which relies on the infinite-time ruin probabilities in the continuous time renewal risk model to derive both an upper and a lower bound for $(\psi^*(u), u \in \mathbb{N})$ within the continuous time renewal risk model with $N^c$ an ordinary renewal process. From now on, we assume that $\lfloor x \rfloor$ represents the largest integer $\leq x$. $Y \overset{d}{=} Z$ indicates that $Y$ is equal in distribution to $Z$ and $Y \overset{d}{\leq} Z$ if the r.v. $Y$ dominates the r.v. $Z$ under the stochastic dominance order (see e.g. Shaked and Shanthikumar (1994)).

Without loss of generality, we consider in this section a continuous time renewal risk model with $c^\ast = 1$. To approximate this continuous time risk model, we consider two specific (time-modified) discrete time renewal risk models. In the first one, the surplus process is defined as $U^\ast = \{U^\ast(t), t \geq 0\}$ and

$$
U^\ast_k = mu - \sum_{i=1}^{N^\ast} (X^c_i - T^\ast_i), \quad k \in \mathbb{N}^+.
$$

(27)

for $m \in \mathbb{N}^+$ where $X^c_i \overset{d}{=} [m X^c_j + 1, T^\ast_j \overset{d}{=} [m T^\ast_j + 2$ and $c = 1$. In the second one, the surplus process is defined as $U^\dagger = \{U^\dagger(t), t \geq 0\}$ and

$$
U^\dagger_k = mu - \sum_{i=1}^{N^\dagger} (X^c_i - T^\dagger_i), \quad k \in \mathbb{N}^+.
$$

(28)

for $m \in \mathbb{N}^+$ where $X^c_i \overset{d}{=} [m X^c_j + 2, T^\dagger_j \overset{d}{=} [m T^\ast_j + 1$ and $c = 1$. Both discrete time renewal risk models are derived from the continuous-one by discretizing the distributions of $X$ and $T$. Finally, we denote by $\psi^\ast$ and $\psi^\dagger$ the infinite-time ruin probability associated respectively to the surplus processes $U^\ast$ and $U^\dagger$.

**Proposition 8.** In the continuous time renewal risk model with $N^c$ an ordinary renewal process and $c^\ast = 1$, the infinite-time ruin probability satisfies

$$
\psi^*(mu) \leq \psi^\ast(u) \leq \psi^\dagger(mu),
$$

(29)

for $u \in \mathbb{N}$ and $\forall m \in \mathbb{N}^+$.
Proof. From (26), one knows that
\[
\psi^l(i) = \Pr(\max_{k \in \{1, \ldots, l\}} \sum_{t=1}^{k} m(X_t - T^*_t) > mu),
\]
(30)
for \(m \in \mathbb{N}^+\) which means that bounds to \(\psi^l\) can be obtained by finding bounds on \(m(X_t - T^*_t)\) under the stochastic dominance order. Since \([mT^*_t] \leq mT^*_t \leq [mT^*_t] + 1\) and \([mX_t] \leq mX_t \leq [mX_t] + 1\), one deduces that
\[
[mX_t] - (([mT^*_t] + 1) \leq mX_t - T^*_t \leq ([mX_t] + 1) - [mT^*_t],
\]
(31)
for \(i \in \mathbb{N}^+\). In order to ultimately use the discrete time renewal risk model (for which it is assumed that waiting times and individual claim amounts can only take positive integer values) to find bounds for \(\psi^l\), we rewrite (31) as
\[
X_t^* - T^*_t \leq m(X_t^* - T^*_t) \leq X_t^* - T^*_t,
\]
(32)
from which one concludes
\[
\sum_{i=1}^{k} (X_t^* - T^*_t) \leq m \sum_{i=1}^{k} (X_t^* - T^*_t) \leq \sum_{i=1}^{k} (X_t^* - T^*_t),
\]
From Theorem 1.2.16 of Müller and Stoyan (2002), it follows
\[
\max_{k \in \{1, \ldots, l\}} \left\{ \sum_{i=1}^{k} (X_t^* - T^*_t) \right\} \leq \max_{k \in \{1, \ldots, l\}} \left\{ \sum_{i=1}^{k} m(X_t^* - T^*_t) \right\} \leq \sum_{i=1}^{k} \max_{k \in \{1, \ldots, l\}} \left\{ X_t^* - T^*_t \right\},
\]
which implies that
\[
\psi^l(mu) \leq \psi^l(u) \leq \psi^l(mu),
\]
(33)
for \(l \in \mathbb{N}^+\) where \(\psi^l, \psi^l\) and \(\psi^l\) are the probability of ruin on or before the \(l\)th claim in their respective risk models. Note that \(c^l = c^l = 1\). Since (33) holds for all \(l,\) we derive (29) by letting \(l \to \infty\) in (33). \(\square\)

With (29), we can theoretically bound \(\psi^l\) by ruin probabilities obtained within some discrete time risk models. As discussed previously, those ruin probabilities cannot be computed exactly. However, we can build genuine upper (lower) bound to \(\psi^l(u)\) by approximating \(\psi^l(mu)\) of (29) by the method proposed in Section 3 and then add (subtract) the error bound of Proposition 3 to the appropriate case. Note that the results of the above proposition can easily be extended to the case where \(X_t^*\) is a delayed renewal process.

Also, one must be careful to choose \(m (m \in \mathbb{N}^+)\) large enough to ensure that the relative security margin \(\eta\) remains positive after the discretization. Note that the larger is \(m\) in Proposition 8, the tighter will be the upper and lower bounds for the infinite-time ruin probabilities. This can be viewed from (31) since the r.v.’s \(X_t^*/m\) or \(X_t^*/m\) and \(T^*_t/m\) or \(T^*_t/m\) converge in distribution to the r.v.’s \(X^*\) and \(T^*\) respectively as \(m\) goes to infinity.

5.3. An application

In this section, we consider a numerical example to illustrate the application of Proposition 8 to approximate the infinite-time ruin probabilities in the continuous time renewal risk model. This application will also allow us to apply the algorithm proposed in Section 3 to compute the infinite-time ruin probabilities in the discrete time renewal risk model. In order to compare our numerical upper and lower bounds, we consider a continuous time renewal risk model for which the infinite-time ruin probabilities have a closed-form expression. Our illustration is based on Example 4 of Drekic et al. (2004), which we briefly recall.

We suppose that \(c^l = 1\). The interclaim time generic r.v. \(T^*\) is assumed to be a mixture of three Erlang distributions with p.d.f.
\[
f_{T^*}(t) = \sum_{k=1}^{3} \frac{k^{t-1} e^{-t}}{(t-1)!},
\]
for \( t > 0 \) where \((v_1, v_2, v_3) = (0.4, 0.2, 0.4)\). The individual claim amounts are distributed according to a feedforward Coxian distribution with p.d.f.

\[
f_{X_c}(x) = 4 \frac{e^{-x}}{5} + 7 \frac{(2e^{-x})}{5} - 8 \frac{(3e^{-x})}{5} + 2 \frac{(4e^{-x})}{5},
\]

for \( x > 0 \) and mean \( E[X_c] = 16/15 \). Based on a numerical procedure described in Asmussen (2000), the following expression for the infinite-time ruin probabilities in the above risk model is derived in Drekic et al. (2004)

\[
\psi(u) = 0.23381e^{-3.6926u} - 0.26698e^{-3.5967u} + 0.04517e^{-1.9065u} + 0.46507e^{-0.5568u}.
\]

(34)

For various initial surplus values, we compare in Fig. 1 the exact values of the infinite-time ruin probabilities obtained with (34) to both lower and upper bounds of Proposition 8 (with \( m = 16, 32 \) and 128).

For the lower and upper bounds of Fig. 1, the error term (20) is lower than \( 5 \times 10^{-7} \) for each value of \( m \) considered using (25) as the upper bound for the ruin probability. The pair \((t_0, a_0)\) used to produce the numerical values was (550, 450), (1100, 900) and (4400, 3600) for \( m = 16, 32 \) and 128, respectively. As expected, the range between the lower and upper bounds decreases as the scaling factor decreases (or \( m \) increases) and both numerical values become close to the exact values provided by (34). To better appreciate our approximation method, we provide, in Table 1, numerical values of the infinite-time ruin probabilities and our numerical upper and lower bounds for \( m = 128 \).

The results indicate that we can have a good idea of the "true" infinite-time ruin probabilities by taking the average of the lower and upper bounds, even for a large scaling factor (even for a large value of \( m \)).

6. Additional results

In this section, we pursue our study of the discrete time renewal risk model by looking at the time of ruin and the severity of ruin. Also, we propose an extension to the discrete time renewal risk model in which the premium rate is no longer constant. We show how easy it is within the framework of this discrete time risk model to extend the methods previously proposed to approximate certain risk measures.

6.1. Severity of ruin

The severity of ruin is another quantitative measure of risk that has been studied within the framework of different risk models. We denote by \( G(u, k) = \Pr(\tau < \infty, |U_\tau| \leq k | U_0 = u) \) the probability that, for a given surplus \( u \), ruin will occur and the deficit at ruin will be less than or equal to \( k \) and its corresponding mass function by \( g(u, k) = \).
Table 1
Numerical upper and lower bounds with \( m = 128 \)

<table>
<thead>
<tr>
<th>( u )</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.4724</td>
<td>0.4799</td>
<td>0.4771</td>
</tr>
<tr>
<td>0.25</td>
<td>0.4138</td>
<td>0.4215</td>
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<td>0.5</td>
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<td>0.3672</td>
<td>0.3642</td>
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<tr>
<td>0.75</td>
<td>0.3113</td>
<td>0.3187</td>
<td>0.3157</td>
</tr>
<tr>
<td>1</td>
<td>0.2692</td>
<td>0.2763</td>
<td>0.2734</td>
</tr>
<tr>
<td>1.25</td>
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<td>0.2395</td>
<td>0.2367</td>
</tr>
<tr>
<td>1.5</td>
<td>0.2015</td>
<td>0.2077</td>
<td>0.2051</td>
</tr>
<tr>
<td>1.75</td>
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<td>0.1803</td>
<td>0.1778</td>
</tr>
<tr>
<td>2</td>
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<td>0.1566</td>
<td>0.1542</td>
</tr>
<tr>
<td>2.5</td>
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<td>0.1183</td>
<td>0.1163</td>
</tr>
<tr>
<td>3</td>
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<td>0.0896</td>
<td>0.0878</td>
</tr>
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<td>0.0679</td>
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</tr>
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</tr>
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<td>0.0390</td>
<td>0.0380</td>
</tr>
<tr>
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<td>0.0296</td>
<td>0.0288</td>
</tr>
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<td>0.0165</td>
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<tr>
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<td>0.0054</td>
</tr>
<tr>
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<td>0.0019</td>
<td>0.0018</td>
</tr>
<tr>
<td>15</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

Pr(\( \tau < \infty \), \( |U_\tau| = k |U_0 = u \)) for \( k \in \mathbb{N}^+ \) and \( u \in \mathbb{N} \). To obtain the defective distribution of the severity of ruin, we propose to use a similar numerical algorithm to the one used for the computation of the infinite-time ruin probabilities. This algorithm relies on the property \( G(u, k) \to 0 \) as \( u \to \infty \) for \( k \in \mathbb{N}^+ \). Therefore, we propose to approximate \( \{G(u, k), u \in \mathbb{N}\} \) by \( G_a(u, k) = 0 \), for \( u = u_0, u_0 + 1, \ldots \) and where \( G_a(u, k) \) for \( u = 0, \ldots, u_0 - 1 \) is obtained by solving the following system of \( u_0 \) equations-\( u_0 \) unknown parameters

\[
G_a(u, k) = \sum_{j=1}^{u_0} f_X(j) \sum_{i=0}^{(u_0+1)/u_0-1} G_a(u+c+j) f_X(u+c+j) - F_X(u+c+j),
\]

for all \( k \in \mathbb{N}^+ \). An error bound similar to the one obtained for the infinite-time ruin probabilities can similarly be found.

**Example.** We again consider the example discussed in Section 5.3. Among the possible approximations to

\[
G^*(u, y) = \Pr(\tau^* < \infty, |U_{\tau^*}^*| \leq y|U_0^* = u),
\]

in the continuous time risk model, we use \( G(u_m, y_m) \) obtained in the framework of the discrete time renewal risk model with interarrival times \( \{T_i, i \in \mathbb{N}^+\} \) and claim amount \( \tau X_i \), \( X_i \in \mathbb{N}^+ \). Other discrete-time risk models using either \( \{T_i, i \in \mathbb{N}^+\} \) or \( \{T_i, i \in \mathbb{N}^+\} \) has a discretized version of the claim amount \( \tau X_i \) could have been used to approximate the continuous-time risk model. In Fig. 2, we depict the approximated and exact values of \( G^*(1, y) \) for \( m = 16, 32, 64 \) (see Drekic et al., 2004 for the exact values). As expected, the approximated values seems to be closer to the exact values of \( G^*(1, y) \) as \( m \) increases.

### 6.2 Premium depending on the current reserve

We extend our discrete time risk model by assuming that the premiums are no longer received at a constant rate. As an example, we assume that the premium rate in period \( k \) is a function of the surplus level at time \( k - 1 \). We denote by \( c(u) \) the premium received in a given period if the surplus level at the beginning of this period is \( u \).
As an illustration, we suppose
\[ c(u) = \begin{cases} 
  c_1, & u = 0, 1, \ldots, n \\
  c_2, & u = n + 1, n + 2, \ldots
\end{cases} \]
where \( c_1 > c_2 \) with \( c_1, c_2 \in \mathbb{N}^+ \) and \( n \in \mathbb{N}^+ \). We also assume \( E[X - c_i T] < 0 \) for \( i = 1, 2 \). For this risk model, the finite-time non-ruin probabilities can be computed recursively with
\[
\phi(u, n) = \sum_{i=1}^{n} f_T(j) \frac{f_X(i) \phi(u + \Pi(u, j) - i, n - j)}{\Pi_1(u, j)} + \bar{F}_T(n),
\]
for \( u \in \mathbb{N}, n \in \mathbb{N}^+ \) where \( \phi(u, 0) = 1 \) and the total premium income received for the first \( j \) periods if the initial surplus level is \( u \), \( \Pi(u, j) \), is defined as
\[
\Pi(u, j) = \begin{cases} 
  c_1 j - (c_1 - c_2) \max \left( j - \left\lfloor \frac{u}{c_1} \right\rfloor, 0 \right), & u < n \\
  c_2 j, & u \geq n
\end{cases}
\]
for \( u \in \mathbb{N} \) and \( j \in \mathbb{N}^+ \).
For the infinite-time ruin probabilities, we take \( n \to \infty \) in (35) and we obtain
\[
\phi(u) = \sum_{j=1}^{\infty} f_T(j) \sum_{i=1}^{u + \Pi(u, j)} f_X(i) \phi(u + \Pi(u, j) - i),
\]
for \( u \in \mathbb{N} \). Using a similar procedure to the one proposed in Section 3, we can approximate \( \phi(u) \) in (36) by \( \phi^a(u) \) where
\[
\phi^a(u) = 1,
\]
for \( u = u_0, u_0 + 1, \ldots \) and \( \phi^a(u) \) for \( u = 0, \ldots, u_0 - 1 \) which is obtained by solving the following system of \( u_0 \) equations-\( u_0 \) unknown parameters
\[
\phi^a(u) = \sum_{j=1}^{u_0} f_T(j) \sum_{i=1}^{u_0 + \Pi(u, j) - 1} f_X(i) \phi(u + \Pi(u, j) - i) + \sum_{j=1}^{u_0} f_T(j) \sum_{i=1}^{\Pi(u, j) - 1} f_X(i) + \bar{F}(u).
\]
In this case, the error bound (9) between \( \{\phi(u), u \in \mathbb{N}\} \) and \( \{\phi^a(u), u \in \mathbb{N}\} \) still holds and can be similarly proven.
The recursive formulas (35) and (36) can easily be extended to the case when the premium rate $c(u)$ can take more than two values. Ruin probabilities obtained in the discrete time renewal risk model can also be used to approximate ruin probabilities in a corresponding continuous time renewal risk model as it will be illustrated next. Note that other cases can be treated similarly as the non-level premium rate example of this section. For example, one may consider a discrete time renewal risk model where the surplus is invested at a fixed rate of interest $r$ at the beginning of each period.

**Example.** Let us assume that $X^c$ is exponentially distributed with mean 1.4 and $T^c$ has a Weibull distribution with $F_{T^c}(t) = 1 - e^{-\theta t/\theta^3}$ for $t > 0$ where $\theta$ is fixed such that $E[T^c] = 2$. The premium rate is no longer level. In this example, we suppose that the premium rate is 1 when the surplus level is higher than 5 and 2, otherwise. To our knowledge, there is no explicit expression for the infinite-time ruin probabilities in this case. Therefore, we rely on the extended discrete time renewal risk models to provide an approximation to this continuous time risk model, along the same lines as we have previously proposed to use discrete time renewal risk models to approximate a continuous time renewal risk model. Here however, instead of finding discrete time renewal risk models that offer an upper and a lower bound to the ruin probabilities in the continuous time risk model, we use a single discrete time renewal risk model obtained through the discretization of the interarrival time and claim amount r.v.’s $X^c$ and $T^c$ on an upper basis (i.e. the discrete r.v.’s $X^*$ and $T^*$, respectively). The approximated values of the ruin probabilities for this discrete-time risk model with scaling factors $h = 1/4, 1/8, 1/16, 1/32$ are shown in Fig. 3.

Given the graph, it seems that the approximated values tend to the exact ones as $h$ becomes smaller. We also observe a change in the curves when the modification in the premium level occurs.

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