Dependent risk models with Archimedean copulas: A computational strategy based on common mixtures and applications

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**A B S T R A C T**

In this paper, we investigate dependent risk models in which the dependence structure is defined by an Archimedean copula. Using such a structure with specific marginals, we derive explicit expressions for the pdf of the aggregated risk and other related quantities. The common mixture representation of Archimedean copulas is at the basis of a computational strategy proposed to find exact or approximated values of the distribution of the sum of risks in a general setup. Such results are then used to investigate risk models in regard to aggregation, capital allocation and ruin problems. An extension to nested Archimedean copulas is also discussed.

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1. Introduction

The present paper deals with risk models incorporating dependent components whose dependence structure is induced via an Archimedean copula. Copula theory provides a flexible approach for modeling the dependency relationship between risks. A copula is a multivariate distribution function for which the marginals are standard uniformly distributed. See e.g. Joe (1997) or Nelsen (2007) for further details. One important class of copulas is the class of Archimedean copulas.

A $d$-dimensional copula $C$ is said to be an Archimedean copula if

$$C(u_1, \ldots, u_d) = \psi \left( \psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d) \right), \quad \text{for} \quad (u_1, \ldots, u_d) \in [0, 1]^d. \quad (1)$$

The continuous and strictly decreasing function $\psi$ is called the generator of the copula, where $\psi: [0, \infty) \to [0, 1], \psi(0) = 1$ and $\lim_{t \to -\infty} \psi(t) = 0$. In the same manner, $\psi^{-1}: [0, 1] \to [0, \infty)$, for which $\psi^{-1}(0) = \inf \{ t : \psi(t) = 0 \}$, where $\psi^{-1}$ is the inverse of the generator $\psi$. McNeil and Nešlehová (2009) show that (1) is a $d$-dimensional copula if and only if $\psi$ is a $d$-monotone function.

In this paper, we consider a specific class of Archimedean copulas with completely monotone generators. By using Bernstein’s theorem (see e.g. Feller, 1971), it has been shown that such generators correspond to the Laplace–Stieltjes Transform (LST) of a strictly positive rv $\Theta$ with cumulative distribution function (cdf) $F_\Theta$, where the LST of the rv $\Theta$ is given by

$$L_\Theta(t) = \int_0^\infty e^{-t \theta} \mathrm{d}F_\Theta(\theta) = E \left[ e^{-t \Theta} \right]. \quad (2)$$

Then, the expression in (1) becomes

$$C(u_1, \ldots, u_d) = L_\Theta \left( L_\Theta^{-1}(u_1) + \cdots + L_\Theta^{-1}(u_d) \right). \quad (3)$$

The strictly positive rv $\Theta$, which can be either discrete or continuous, corresponds to a latent mixing rv, and there is a one-to-one relation between its distribution and the expression of an Archimedean copula $C$. This special class of Archimedean copulas defined in (3) is intimately linked to common mixtures. As explained in Denuit et al. (2006) and Embrechts et al. (2005), the representation of an Archimedean copula $C$ as a common mixture allows us to identify the conditional univariate cumulative distribution functions (cdfs) of $(U_i | \Theta = \theta), \ldots, (U_i | \Theta = \theta)$, where $U_i = (U_1, \ldots, U_d)$ and $U_i \sim \text{Unif} (0, 1), i = 1, 2, \ldots, d$. Using the definition in (2) in combination with (3), we have the following representation of an Archimedean copula $C$ as a common mixture:

$$C(u_1, \ldots, u_d) = \psi \left( \psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d) \right)$$

$$= \psi \left( L_\Theta^{-1}(u_1) + \cdots + L_\Theta^{-1}(u_d) \right)$$

$$= \int_0^\infty \prod_{i=1}^d e^{-\theta L_\Theta^{-1}(u_i)} \mathrm{d}F_\Theta(\theta). \quad (4)$$
which becomes
\[ C(u_1, \ldots, u_d) = F_{U_i}^d(u_1, \ldots, u_d) = \int_0^\infty \prod_{i=1}^d F_{U_i|\Theta = \theta}(u_i) \, dF_\Theta(\theta). \] (4)

Then, (4) implies that the conditional cdf of \((U_i|\Theta = \theta)\) is
\[ F_{U_i|\Theta = \theta}(u_i) = e^{-\theta \mathcal{L}_{\theta}^{(i)}(u_i)}, \]
for \(i = 1, \ldots, d, u_i \in [0, 1], \theta > 0\), which leads to the useful representation of the copula \(C\) as a common mixture. Examples of the most popular Archimedean copulas (\(\Theta\) discrete and continuous) are provided in Appendix section (see e.g., Nelsen, 2007 for an extensive list of Archimedean copulas).

The above representation provides a natural sampling algorithm (see e.g., Marshall and Olkin, 1988; Hofert, 2008 and references therein) useful for actuarial science and quantitative risk management purposes. Mixture representations are frequently involved in risk models. For example, in a credit-risk context, Vasicek (1987) was the first to develop the idea of the conditional independence of all defaults upon some market factor. Later, this idea was used by Li (2000), Schönbucher and Schubert (2001), Frey and McNeil (2001) and put into a copula setup. Among the papers that exploit the conditional independence technique with Archimedean and nested Archimedean copulas in a credit risk context one finds for example Schönbucher (2002) and Hofert and Scherer (2011). Copulas in a credit risk setting are also treated with much detail in Cherubini et al. (2004). See references therein. Several researchers have also used the conditional representation of Archimedean and nested Archimedean copulas to derive their sampling algorithms see e.g. Marshall and Olkin (1988), McNeil (2008) and Hofert (2012). Archimedean risk models are also involved in the context of collective risk models with dependence. Thanks to the latter representation, Albrecher et al. (2011) were able to establish explicit formulas for the ruin probability with dependence among claim sizes and among claim inter-occurrence times modeled by Archimedean copulas. Also, a recent work of Jordanova et al. (2017), considered dependent inter-arrival times and exploited Archimedean copulas as treated in Albrecher et al. (2011).

Our objective here is to explore in more depth the mixture representation of an Archimedean copula and its advantages. More precisely, we propose a new strategy relying on (4), to tackle risk assessment problems such as risk aggregation for finite and random sums, capital allocation, ruin problems and so on. We show that this representation allows to avoid entirely or partially Monte Carlo (MC) simulations. Furthermore, this methodology is accurate, exact in specific cases, very flexible, and most importantly, naturally applicable in high dimensions.

The outline of the paper is as follows. We propose in Section 2, a computational methodology based on the common mixture representation of Archimedean copulas in different settings. This strategy allows to derive the distribution of the aggregated risks which can be later used in different applications. Analytic expressions related to the aggregated risks are also derived for some special cases using specific marginal distributions. Section 3 deals with capital allocation issues involving the strategy proposed in Section 2. Random sums are then considered in Section 4. Sections 5 and 6 are devoted to the investigation of ruin problems. Finally, Section 7 discusses the application of the mixture-based strategy in the case of a hierarchical dependence structure based on Archimedean copulas.

2. Computational methodology based on the common mixture representation of Archimedean copulas

2.1. Common mixture representation

Let \(X = (X_1, \ldots, X_d)\) be a vector of rvs with multivariate distribution defined in terms of a \(d\)-dimensional Archimedean copula \(C\) given in (3). The multivariate cdf \(F_X\) of \(X\) can be defined with the copula \(C\) and marginal univariate cdfs \(F_{X_i}\) as
\[ F_X(x_1, \ldots, x_d) = C(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)) . \] (5)

The common mixture representation of \(F_X\) is given by
\[ F_X(x_1, \ldots, x_d) = \int_0^\infty \prod_{i=1}^d F_{X_i|\Theta = \theta}(x_i) \, dF_\Theta(\theta), \] (6)

where
\[ F_{X_i|\Theta = \theta}(x_i) = e^{-\theta \mathcal{L}_{\theta}^{(i)}(F_{X_i}(x_i))} \quad (i = 1, 2, \ldots, d). \] (7)

Similarly, we can define the multivariate distribution of \(X\) through its multivariate survival function with the copula \(C\) and marginal univariate survival functions \(F_{X_1}, \ldots, F_{X_d}\), i.e.,
\[ F_{X}(x_1, \ldots, x_d) = \int_0^\infty \prod_{i=1}^d e^{-\theta \mathcal{L}_{\theta}^{(i)}(1-F_{X_i}(x_i))} \, dF_\Theta(\theta). \] (8)

Then, the common mixture representation of \(F_{X}\) is given by
\[ F_{X}(x_1, \ldots, x_d) = \int_0^\infty \prod_{i=1}^d e^{-\theta \mathcal{L}_{\theta}^{(i)}(1-F_{X_i}(x_i))} \, dF_\Theta(\theta) \] (9)

where
\[ F_{X_i|\Theta = \theta}(x_i) = e^{-\theta \mathcal{L}_{\theta}^{(i)}(1-F_{X_i}(x_i))} \quad (i = 1, 2, \ldots, d). \] (10)

Our main objective in what follows is to find \(E[\phi(S)]\), for any integrable univariate function \(\phi\) of the rv \(S = X_1 + \ldots + X_d\), or to find \(E[\phi(X_1, \ldots, X_d)]\), for any integrable \(d\)-variate function \(\phi\). The required steps to derive these quantities are as follows:

1. Find the expressions of the conditional cdfs (7) or the conditional survival functions (10) of \(X_1|\Theta = \theta\).
2. Derive the conditional expectations \(E[\phi(S)|\Theta = \theta]\) or \(E[\phi(X_1, \ldots, X_d)|\Theta = \theta]\).
3. Find \(E[\phi(S)]\) and \(E[\phi(X_1, \ldots, X_d)]\) with
\[ E[\phi(S)] = E_\Theta[E[\phi(S)|\Theta = \theta]] = \int_0^\infty E[\phi(S)|\Theta = \theta] \, dF_\Theta(\theta), \]
and
\[ E[\phi(X_1, \ldots, X_d)] = E_\Theta[E[\phi(X_1, \ldots, X_d)|\Theta = \theta]] \]
\[ = \int_0^\infty E[\phi(X_1, \ldots, X_d)|\Theta = \theta] \, dF_\Theta(\theta), \]
where \(S|\Theta = \theta) = \sum_{\ell=1}^d X_\ell|\Theta = \theta\).

We present below three examples with specific marginals but any Archimedean copula with generator \(\mathcal{L}_{\theta}\) in which the required steps just detailed to find the pdf of \(S\) are easily performed and allow explicit expressions.

Example 1. Let \(X = (X_1, \ldots, X_d)\) be a vector of exchangeable Bernoulli rvs where \(X_i \sim Bern(q)\) (\(i = 1, \ldots, d\)) and
\[ F_X(x_1, \ldots, x_d) = C(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)), \]
for \(k_1, \ldots, k_d \in [0, 1]\). From (7), we find that
\[ (X_i|\Theta = \theta) \sim Bern\left(1 - e^{-\theta \mathcal{L}_{\theta}^{(i)}(1-q)}\right), \]
for \( i = 1, 2, \ldots, d \). Therefore, \((S|\theta = \theta)\) follows a binomial distribution with

\[
\begin{align*}
f_S(\theta \rightarrow \theta) (k) &= \binom{d}{k} \left(1 - e^{-\theta \mathcal{L}_{\theta}} \right)^k (1 - \theta \mathcal{L}_{\theta})^{d-k} \\
&= \binom{d}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \theta^{-\mathcal{L}_{\theta}} (1 - \theta \mathcal{L}_{\theta})^{j+d-k},
\end{align*}
\]

and we conclude

\[
f_S(k) = \left(d \right) \sum_{j=0}^{k} \binom{k}{j} (-1)^j \theta^{-\mathcal{L}_{\theta}} (1 - \theta \mathcal{L}_{\theta})^{j+d-k}.
\]

for \( k = 0, 1, 2, \ldots, d \).

**Example 2.** Let \( X = (X_1, \ldots, X_d) \) be a vector of exchangeable Bernoulli rvs where \( X_i \sim \text{Bern}(q) \) \((i = 1, \ldots, d)\) and \( F_X(x_1, \ldots, x_d) = C \left( \prod_{i=1}^{d} F_{X_i}(x_i) \right) \) for \( k_1, \ldots, k_d \in [0, 1) \). It follows from (10) that

\[
(X_i|\theta = \theta) \sim \text{Bern} \left( e^{-\theta \mathcal{L}_{\theta}} \right)
\]

for \( i = 1, 2, \ldots, d \). Since

\[
f_{X_i|\theta \rightarrow \theta} (k) = \left(d \right) \sum_{j=0}^{k} \binom{k}{j} (-1)^j \theta^{-\mathcal{L}_{\theta}} \left(1 - \theta \mathcal{L}_{\theta} \right)^{j+d-k},
\]

we obtain

\[
f_S(k) = \left(d \right) \sum_{j=0}^{k} \binom{k}{j} (-1)^j \theta^{-\mathcal{L}_{\theta}} \left(1 - \theta \mathcal{L}_{\theta} \right)^{j+d-k},
\]

for \( k = 0, 1, 2, \ldots, d \). This expression can be found e.g., in Cossette et al. (2002) or on page 321 in Marceau (2013).

**Example 3.** Let \( \Theta \) be a strictly positive rv with LST \( \mathcal{L}_{\theta} \). Given \( \Theta = \theta \), let \((X_1|\theta = \theta), \ldots, (X_d|\theta = \theta)\) conditionally independent rvs where \((X_i|\theta = \theta) \sim \text{Exp}(\lambda)\) with geometric distribution

\[
f_{x_i|\theta \rightarrow \theta} (k) = \text{Pr} (X_i = k|\theta = \theta) = e^{-\lambda k} (1 - e^{-\lambda r})
\]

and

\[
f_{x_i|\theta \rightarrow \theta} (k) = \text{Pr} (X_i > k|\theta = \theta) = e^{-\lambda (k+1)} (i = 1, 2, \ldots, n)
\]

for \( k = 0, 1, 2, \ldots, n \). Then, \( X = (X_1, \ldots, X_d) \) follows a multivariate mixed geometric distribution where

\[
F_X(x_1, \ldots, x_d) = \mathcal{L}_{\theta} \left( r (k_1 + 1) + \cdots + r (k_n + 1) \right)
\]

with

\[
F_X(x_i) = \mathcal{L}_{\theta} \left( r (k_i + 1) \right) \text{ for } i = 1, 2, \ldots, n.
\]

It implies that

\[
F_X(x_1, \ldots, x_d) = \mathcal{L}_{\theta} \left( \mathcal{L}_{\theta}^{-1} (F_X(x_1)) + \cdots + \mathcal{L}_{\theta}^{-1} (F_X(x_n)) \right)
\]

where \( C \) is an Archimedean copula defined with LST \( \mathcal{L}_{\theta} \).

Now, we are in position to derive the closed-form expression of the pmf of \( S = \sum_{i=1}^{d} X_i \). Observe that \((S_i|\theta = \theta)\) follows a negative binomial distribution with

\[
f_{X_i|\theta \rightarrow \theta} (k) = \text{Pr} (X_i = k|\theta = \theta) = \binom{k+n-1}{k} \theta^{-r} \left(1 - \theta^{-r} \right)^n
\]

for \( k = 0, 1, 2, \ldots, n \). Then, it follows that

\[
f_S(k) = \binom{k+n-1}{k} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \theta^{-r(j+k)},
\]

for \( k = 0, 1, 2, \ldots \). Then, it follows that

\[
f_S(k) = \binom{k+n-1}{k} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \mathcal{L}_{\theta} (r (j + k)),
\]

for \( k = 0, 1, 2, \ldots \).

In the following subsection, we consider the family of multivariate mixed exponential distributions which is equivalent to choosing specific combinations of Archimedean copulas and marginals. For this class of multivariate distributions, we derive analytic expressions for the pdf of the sum of risks and other related quantities of interest.

### 2.2. Closed-form expressions for multivariate mixed exponential distributions

Let \( X \) follow a multivariate mixed exponential distribution which belongs to the class of multivariate distributions constructed by common frailty as explained in e.g. Marshall and Olkin (1988). Briefly, given \( \Theta = \theta \), the conditional distribution of the \( X_i \) is exponential with mean \( \frac{1}{\lambda_i} \), i.e.,

\[
F_{X_i|\theta \rightarrow \theta} (x_i) = e^{-\frac{x_i}{\lambda_i}},
\]

for \( i = 1, \ldots, d \). It implies that the marginal survival function of \( X_i \) is given by

\[
F_{X_i}(x_i) = \mathcal{L}_{\theta} \left( \frac{x_i}{\lambda_i} \right),
\]

for \( i = 1, \ldots, d \). Also, the multivariate survival function of \( X \) is given by

\[
F_X(x_1, \ldots, x_d) = \mathcal{L}_{\theta} \left( \frac{x_1}{\lambda_1} + \cdots + \frac{x_d}{\lambda_d} \right),
\]

which implies that \( F_X \) satisfies (8) with Archimedean copula \( C \) given in (3). Clearly, in this setting, the required steps 1, 2, and 3 of the methodology described in Section 2.1 are easily performed. The expression in (10) is clearly given in (11). We first consider the case where \( 0 < \lambda_1 < \cdots < \lambda_d \). Here, \((S|\Theta = \theta)\) follows a generalized Erlang distribution with pdf

\[
f_S(x) = \frac{\lambda_1}{\lambda_1 - \lambda_2} \left( \prod_{j=1}^{d} \frac{\lambda_j}{\lambda_j - \lambda_{j+1}} \right) \theta^{-\lambda_1} \left(1 - \theta^{-\lambda_1} \right)^{x-1}.
\]

Then, using (13), the unconditional pdf of \( S \) is given by

\[
f_S(x) = \int_{0}^{\infty} f_{S|\Theta \rightarrow \Theta} (x|\Theta) dF_{\Theta}(\Theta)
\]

for \( \Theta \) follows an Erlang distribution with

\[
f_S(x) = \frac{\lambda_1^d e^{-\lambda_1 x}}{\Gamma(d)}
\]

for \( d = 2 \). The bivariate case \((d = 2)\) is detailed on page 295 of Marceau (2013). Sarabia et al. (2017) consider the subclass of multivariate mixed exponential distributions in which \( \lambda_1 = \cdots = \lambda_d = 1 \). In such a case, \((S|\Theta = \theta)\) follows an Erlang distribution with

\[
f_S(x) = \frac{x^{d-1} e^{-x}}{\Gamma(d)}
\]

Sarabia et al. (2017) find the following closed-form expressions for the pdf and the survival function of the rv \( S \):

\[
f_S(x) = \frac{x^{d-1} e^{-x}}{\Gamma(d)}
\]

for \( k = 0, 1, 2, \ldots \).
and
\[ \Pr(S > x) = \sum_{k=0}^{d-1} \frac{x^k}{k!} \left( -1 \right)^k \frac{d^k}{dx^k} \mathcal{L}_\omega(x) \], \quad x \in \mathbb{R}^+.

From (14), one can deduce the following expression for the TVaR of the rv S:
\[ \text{TVaR}_S(x) = \frac{E \left[ S \mid S \geq \text{VaR}_x(S) \right]}{1 - \kappa} = \frac{d \times (\text{Var}_S)_{\downarrow} 1}{J! \times (1 - \kappa)} \left\{ (-1)^{j-1} \frac{d^{j-1}}{dx^{j-1}} \mathcal{L}_\omega(x) \right\}_{x=\text{VaR}_x(S)}, \]
where \( \text{Var}_S \) is the solution of \( F_S(x) = \kappa \), with \( \kappa \in [0, 1) \). The stop-loss function of S can also be expressed in terms of \( \mathcal{L}_\omega \) as follows
\[ \Pi_\kappa(y) = E(\max(S - y, 0)) = d \sum_{j=0}^{d-1} \frac{y^j}{j!} \left\{ (-1)^{j-1} \frac{d^{j-1}}{dx^{j-1}} \mathcal{L}_\omega(x) \right\}_{x=y} - y \sum_{j=0}^{d-1} \frac{y^j}{j!} \left\{ (-1)^{j-1} \frac{d^{j-1}}{dx^{j-1}} \mathcal{L}_\omega(x) \right\}_{x=y}, \quad y \in \mathbb{R}^+.

Sarabia et al. (2017) give explicit formulas for the pdf of S for some multivariate mixed exponential distributions for which the dependence structure is an Archimedean copula with continuous common factor \( \theta \) such as the Clayton and Gumbel copulas. In the same context, Dacorogna et al. (2016) use similar explicit formulas to compute analytically risk measures and the associated diversification benefit.

Since there are several much used copulas generated from a discrete mixing rv \( \theta \) such as the Ali–Mikhail–Haq (AMH) copula, the Frank copula and the Joe copula, we provide the two following examples to complete those provided in Sarabia et al. (2017). Also, since the derivatives of the generators are known for different Archimedean copulas with discrete or continuous mixing rv (see, e.g., Holert et al., 2012), the results provided in the following examples follow directly from the definitions just presented above.

**Example 4.** Let \( X = (X_1, \ldots, X_d) \) follow a multivariate mixed exponential–geometric distribution, i.e., \( \theta \sim \text{Geo}(q) \) with probability mass function (pmf) \( f_{\theta}(k) = q(1-q)^{k-1}, \) for \( k \in \mathbb{N} \). Clearly, the dependence structure underlying this multivariate distribution is the AMH copula with dependence parameter \( \alpha = 1 - q \). Then, the following properties hold:
1. \( F_{\theta_i}(x) = \mathcal{L}_\omega(x) = \frac{q}{\alpha \left( 1 - q \right)}, \quad x > 0, \quad i = 1, \ldots, d. \)
2. \( F_S(x) = \frac{\exp\left( \sum_{k=1}^{\infty} q^k \right)}{(1-q)^{\alpha}} \), \( x > 0. \)
3. \( f_S(x) = \frac{x^{d-1}}{(1-q)^{\alpha} \Gamma(d)} \) \( \text{Li}_d \left( (1 - q)x^{-\alpha} \right), \) \( x > 0. \)
4. \( \Pr(S > x) = \sum_{k=0}^{d-1} \frac{x^k}{(1-q)^{\alpha} \Gamma(d)} \text{Li}_k \left( (1 - q)x^{-\alpha} \right), \) \( x > 0. \)
5. \( \text{TVaR}_S(x) = \sum_{j=0}^{d-1} \frac{x^j}{(1-q)^{\alpha} \Gamma(d)} \text{Li}_{d-j} \left( (1 - q)x^{-\alpha} \right), \) \( j = 0, \ldots, d. \)
6. \( \Pi_\kappa(y) = -d \sum_{j=0}^{d-1} \frac{y^j}{(1-q)^{\alpha} \Gamma(d)} \text{Li}_{d-j} \left( (1 - q)x^{-\alpha} \right) - y \sum_{j=0}^{d-1} \frac{y^j}{(1-q)^{\alpha} \Gamma(d)} \text{Li}_{d-j} \left( (1 - q)x^{-\alpha} \right), \) \( y \in \mathbb{R}^+. \)

In this subsection, we have discussed multivariate mixed exponential distributions which, as highlighted in the examples, require specific combinations of the mixing rv \( \theta \) and the marginal distributions of \( X_i \) (\( i = 1, \ldots, d \)). The dependence structure defined by the Archimedean copula is governed by \( \theta \) and hence only one choice of marginal distribution can be linked through such a dependency framework. This is very limiting since it does not allow to choose the dependence construction without any regard to the specification of the marginals. This is somewhat counterintuitive with copulas being used as flexible tools to build dependence risk models.

In the next three subsections, we provide insight on how to benefit from the mixture representation of an Archimedean copula for any multivariate distribution defined through an Archimedean copula with a discrete mixing rv \( \theta \) and discrete marginals. Furthermore, we provide a strategy that can be used when one or both of these rvs are continuous.

### 2.3. Discrete mixing rv \( \theta \) and discrete marginals

Our strategy is to use the conditional independence assumption to identify the conditional distribution of \( X_i \) through (7) or (10). This step is usually more difficult for continuous rvs \( X_i \) than for discrete ones which is the basis of the computational strategy presented in Section 2.3. We have recourse to discretization methods in the continuous case. The conditional distributions of \( S \) given \( \theta = \theta_i \) are also easier to identify for discrete distributions.

The proposed strategy has many advantages. First, it is easy to implement regardless of the portfolio’s dimension. Second, it yields the exact values of \( F_S \) for discrete risks \( X_i, i = 1, \ldots, d \) and it gives an accurate approximation for continuous ones. The proposed strategy can also be used in the context of many actuarial risk models involving Archimedean dependence which will be discussed in Sections 4–6.

Our strategy is to make use of the conditional independence representation as well as the convolution of independent rvs to derive a simple computation approach for the distribution of \( S \) and any integrable function of \( X \). Given these techniques, risk aggregation problems for a portfolio of dependent risks with a multivariate joint distribution defined with an Archimedean copula or a nested Archimedean copula, as well as other related quantities become easier to deal with.

Let \( X = (X_1, \ldots, X_d) \) be a vector of discrete rvs such that \( X_i \in A = \{ 0, 1, h, 2h, \ldots \} \) (\( i = 1, \ldots, d \)). The univariate pmf of \( X_i \), the univariate cdf of \( X_i \), the multivariate pmf of \( X \), and the multivariate cdf of \( X \) are respectively denoted by \( f_{X_i}(k), \) \( F_{X_i}(k) = \Pr(X_i = k), \) \( f_X(k), \) \( F_X(k) = \Pr(X \leq k) \),
\[ f_X(k_1, \ldots, k_d) = \Pr(X_1 = k_1, \ldots, X_d = k_d), \]
and
\[ F_X(k_1, \ldots, k_d) = \Pr(X_1 \leq k_1, \ldots, X_d \leq k_d). \]
for \( h > 0 \) and \( k_1, \ldots, k_d \in \mathbb{N}_0 \) where \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). In theory, exact values of \( f_h \) and \( E[\psi(X_1, \ldots, X_d)] \) can be found with

\[
f_h(k) = \sum_{k_1=0}^{k} \cdots \sum_{k_d=0}^{k} \frac{(-1)^{k_1+\cdots+k_d}}{(k_1+\cdots+k_d)!} \times f_{X_k}(k_1h, \ldots, k_dh) \left( k_1 \ldots k_d \right) h \tag{15}
\]

and

\[
E[\psi(X_1, \ldots, X_d)] = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \psi(k_1h, \ldots, k_dh) \times f_X(k_1h, \ldots, k_dh) \tag{16}
\]

where

\[
f_X(k_1h, \ldots, k_dh) = \sum_{i_1=0}^{k_1} \cdots \sum_{i_d=0}^{k_d} (-1)^{i_1+\cdots+i_d} \times f_{X_1}(k_1-i_1h, \ldots, k_d-i_dh) h \tag{17}
\]

The computation of (15), (16), and (17) is feasible when the dimension \( d \) of \( X \) is small (e.g. \( d = 2 \) or 4). However, it rapidly becomes impracticable when \( d \) gets larger.

Assume that the dependence structure of \( X \) is induced via an Archimedean copula \( C \). In such a case, the multivariate cdf of \( X \) (or its survival function) is defined with the copula \( C \) and the univariate cdfs (or the univariate survival functions) of \( X_1, \ldots, X_d \), i.e., (5) and (8) become

\[
F_X(k_1h, \ldots, k_dh) = C \left( F_{X_1}(k_1h), \ldots, F_{X_d}(k_dh) \right) \tag{18}
\]

or

\[
\bar{F}_X(k_1h, \ldots, k_dh) = C \left( \bar{F}_{X_1}(k_1h), \ldots, \bar{F}_{X_d}(k_dh) \right) \tag{19}
\]

for \( k_i \in \mathbb{N}_0 \) and \( i = 1, \ldots, d \).

In the following, let the copula \( C \) in either (18) or (19) be an Archimedean copula defined by a discrete mixing rv \( \Theta \) such that \( E[\theta] \) is finite. Then, we can take advantage of the common mixture representation to compute the exact values of \( f_h \) and \( E[\psi(X_1, \ldots, X_d)] \).

Assuming discrete marginals and a discrete mixing rv \( \Theta \) and using (6), the expression for \( F_X \) in (18) becomes

\[
F_X(k_1h, \ldots, k_dh) = \sum_{\theta=1}^{\infty} \prod_{i=1}^{d} f_{X_i}(\theta)(k_i) f_{\Theta}(\theta) \tag{20}
\]

and, from (7), we have

\[
f_{X_i}(\theta)(k_i) = e^{-\theta \psi(1)} \left( f_{X_i(k_ih)}(\theta) \right) \tag{21}
\]

for \( k_i \in \mathbb{N}_0, i = 1, 2, \ldots, d \), and \( \theta \in \mathbb{N} \). For \( i = 1, 2, \ldots, d \) and for each \( \theta \in \mathbb{N} \), we can easily find the values of \( f_{X_i}(\theta)(k_i) \) with

\[
f_{X_i}(\theta)(k_i) = \left\{ \begin{array}{ll}
e^{-\theta \psi(1)}(f_{X_i(0)}(\theta)) & , k_i = 0 \\
e^{-\theta \psi(1)}(f_{X_i(k_i\theta)}(\theta)) - e^{-\theta \psi(1)}(f_{X_i(k_i\theta-1)}(\theta)), & k_i \in \mathbb{N}.
\end{array} \right. \tag{22}
\]

Similarly, using (9), the expression for \( \bar{F}_X \) in (19) turns into

\[
\bar{F}_X(k_1h, \ldots, k_dh) = \sum_{\theta=1}^{\infty} \prod_{i=1}^{d} \bar{F}_{X_i}(\theta)(k_i) f_{\Theta}(\theta) \tag{23}
\]

and, from (10), we have

\[
\bar{F}_{X_i}(\theta)(k_i) = e^{-\theta \psi(1)} \left( \bar{F}_{X_i(k_i\theta)}(\theta) \right) \tag{24}
\]

for \( k_i \in \mathbb{N}_0, i = 1, 2, \ldots, d \), and \( \theta \in \mathbb{N} \). It follows that

\[
f_{X_i}(\theta)(k_i) = \left\{ \begin{array}{ll}
e^{-\theta \psi(1)}(f_{X_i(0)}(\theta)) & , k_i = 0 \\
e^{-\theta \psi(1)}(f_{X_i(k_i\theta-1)}(\theta)) - e^{-\theta \psi(1)}(f_{X_i(k_i\theta)}(\theta)), & k_i \in \mathbb{N}.
\end{array} \right. \tag{25}
\]

for \( i = 1, 2, \ldots, d \) and for each \( \theta \in \mathbb{N} \).

Then, either (22) or (25), the expression for \( f_X(k_1h, \ldots, k_dh) \) is now given by

\[
f_X(k_1h, \ldots, k_dh) = \sum_{\theta=1}^{\infty} \prod_{i=1}^{d} f_{X_i}(\theta)(k_i) f_{\Theta}(\theta) \tag{26}
\]

Let \( S(\theta) = \sum_{i=1}^{d} (X_i| \theta) = \theta \) be the sum of conditionally independent rvs and \( f_{S}(\theta) \) be the corresponding pmf. Due to the representation of \( f_X \) in (26), the expression for the pmf of \( S \) can be written as follows

\[
f_S(k) = \sum_{\theta=1}^{\infty} f_{S}(\theta)(k) f_{\Theta}(\theta), \quad k \in \mathbb{N}_0. \tag{27}
\]

Since \( f_{S}(\theta) \) corresponds to the convolution product of \( f_{X_1}(\theta), \ldots, f_{X_d}(\theta) \), traditional aggregation tools (e.g. FFT or DePril’s algorithm) see e.g. Panjer et al. (2008) from actuarial science can be applied to find values of \( f_{S}(\theta) \) for each \( \theta \). It is important to note here that our strategy leads to exact values of \( f_X \) contrarily to MC simulation methods. This is also true even when \( n \) is large.

Given the representation of \( f_X \) in (26), \( E[\psi(X_1, \ldots, X_d)] \) becomes

\[
E[\psi(X_1, \ldots, X_d)| \Theta = \theta] \tag{28}
\]

where

\[
E[\psi(X_1, \ldots, X_d)| \Theta = \theta] = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \psi(k_1h, \ldots, k_dh) \prod_{i=1}^{d} f_{X_i}(\theta)(k_i). \tag{29}
\]

Applications of (28) and (29) are provided in Section 3.

The procedure to compute the values of \( f_X(k) \) is summarized in the following algorithm.

**Algorithm 6. Computation of the values of \( f_X(k) \).**

1. Let \( \theta^* \) be chosen such that \( f_{\Theta}(\theta^*) \leq \theta \) where \( \theta \) is fixed as small as desired (e.g. \( \theta = 10^{-10} \)).
2. Fix \( \theta = 1 \).
3. For \( i = 1, \ldots, d \), calculate either \( f_{X_i}(\theta)(k_i) \) with (21) or \( f_{X_i}(\theta)(k_i) \) with (24), for \( k_i \in \mathbb{N}_0 \).
4. For \( i = 1, \ldots, d \), calculate \( f_{S}(\theta)(k_i) \) with either (22) or (25).
5. Using e.g. FFT or DePril’s Algorithm, compute \( f_{S}(\theta)(k) \) for \( k \in \mathbb{N}_0 \).
6. Repeat steps 3, 4, and 5 for \( \theta = 2, \ldots, \theta^* \) where \( \theta^* \) is chosen such that \( f_{\Theta}(\theta^*) \leq \theta \) where \( \theta \) is fixed as small as desired (e.g. \( \theta = 10^{-10} \)).
7. Compute \( f_X(k) = \sum_{\theta=1}^{\theta^*} f_{S}(\theta)(k) f_{\Theta}(\theta), \) for \( k \in \mathbb{N}_0 \).

Some remarks must be made in regard to the proposed methodology of this section and the tail dependence phenomenon of an Archimedean copula \( C \). Let us recall that if the lower and upper-tail dependence coefficients exist for an Archimedean copula \( C \) with mixing rv \( \Theta \), then according to Joe and Hu (1996), these coefficients \( \kappa_L \) and \( \kappa_U \) can be written in terms of \( \psi \) as follows

\[
\kappa_L = \lim_{t \to \infty} \frac{L_{\psi}(2t)}{L_{\psi}(t)} = \lim_{t \to \infty} \frac{\psi(2t)}{\psi(t)}
\]
Table 1
Values of the expectation, variance, VaR and TVaR of \( S = X_1 + \cdots + X_4 \) where \( F_{X_1,\ldots,X_4} \) is defined with the Frank copula.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \alpha = 1 )</th>
<th>( \alpha = 3 )</th>
<th>( \alpha = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E[S] )</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>( \text{Var}(S) )</td>
<td>9.99256</td>
<td>15.15425</td>
<td>19.90096</td>
</tr>
<tr>
<td>( \text{VaR}_{0.05}(S) )</td>
<td>14</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>( \text{VaR}_{0.09}(S) )</td>
<td>20</td>
<td>21</td>
<td>23</td>
</tr>
<tr>
<td>( \text{TVaR}_{0.09}(S) )</td>
<td>15.82355</td>
<td>17.11038</td>
<td>18.04888</td>
</tr>
<tr>
<td>( \text{TVaR}_{0.99}(S) )</td>
<td>20.88055</td>
<td>22.39553</td>
<td>23.41423</td>
</tr>
</tbody>
</table>

Table 2
Values of the pmf of \( S = X_1 + \cdots + X_4 \) where \( F_{X_1,\ldots,X_4} \) is defined with the Frank copula.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( s )</th>
<th>Exact values of ( f_\alpha(s) ) (with (15))</th>
<th>Exact values of ( f_\alpha(s) ) (Algorithm 6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.992354e-05</td>
<td>1.992354e-05</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>0.03925912</td>
<td>0.03925912</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>0.1199567</td>
<td>0.1199567</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>0.0001013881</td>
<td>0.0001013881</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>0.0087858345</td>
<td>0.0087858345</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>0.0003887516</td>
<td>0.0003887516</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>0.0091872490</td>
<td>0.0091872490</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>0.0767524930</td>
<td>0.0767524930</td>
</tr>
</tbody>
</table>

Table 3
Values of the expectation, variance, VaR and TVaR of \( S = X_1 + \cdots + X_100 \) where \( F_{X_{i1,\ldots,X_{100}}} \) is defined with the AMH copula.

\[
\begin{array}{|c|c|c|c|}
\hline
\alpha & 0 \text{ (independence)} & 0.5 & 0.9 \\
\hline
E[S] & 100 & 100 & 100 \\
\text{Var}(S) & 1454.027 & 2793.690 & 2793.690 \\
\text{VaR}_{0.05}(S) & 156 & 172 & 172 \\
\text{VaR}_{0.09}(S) & 225 & 242 & 242 \\
\text{TVaR}_{0.09}(S) & 176.206 & 192.651 & 192.651 \\
\text{TVaR}_{0.99}(S) & 233.651 & 250.590 & 250.590 \\
\hline
\end{array}
\]

and

\[
\lambda_{ij} = 2 - \lim_{t \to 0} \frac{1 - \mathbb{L}_{ij}(2t)}{1 - \mathbb{L}_{ij}(t)} = 2 - 2\lim_{t \to 0} \frac{\mathbb{L}_{ij}(2t)}{\mathbb{L}_{ij}(t)}.
\]

For the special case of discrete mixing rv \( \Theta \), the underlying Archimedean copulas cannot have lower tail dependence, i.e. \( \lambda_{ij} = 0 \). This implies that our proposed methodology does not allow lower tail dependence. Also, if \( E[\Theta] \) is finite, then \( \lambda_{ij} = 0 \) (see Hofert, 2010 page 62 for proof). This means that it is possible to find a finite \( \theta^* \) for which the methodology described in Algorithm 6 leads to exact values of the pmf of the rv \( S \). In such a case, the methodology works well for notably the AMH and Frank copulas. Note that the proposed methodology cannot be applied when \( E[\Theta] \) is infinite (e.g. for Joe’s copula) since Algorithm 6 suggests to truncate the distribution of \( \Theta \) at \( \theta^* \). Such a truncation leads to an Archimedean copula with no upper tail dependence which violates the initial assumption.

In the following two examples, we illustrate the accuracy of our computational methodology. More precisely, we present a first example which considers a small portfolio. This allows us to compare the values of \( f_\alpha \) obtained with Algorithm 6 and (15). As expected, both results coincide. The second example is somewhat similar but illustrates the applicability of Algorithm 6 to large portfolios.

Example 7. Let \( F_X \) be defined as in (18) with the Frank copula as given in Appendix and \( X_i \sim \text{Bin}(10, q_i) \), where \( q_i = 0.1i \), for \( i = 1, 2, 3, 4 \). In this case, the mixing rv \( \Theta \) follows a logarithmic distribution. While values of \( E[S], \text{Var}(S), \text{VaR}(S) \) and \( \text{TVaR}(S) \) are given in Table 1, Table 2 provides the exact values of \( f_\alpha \) obtained with (15) and Algorithm 6.

Example 8. Let \( X_i \sim \text{Bin}(10, q_i) \), with \( q_i = 0.1 \), for \( i = 1, \ldots, 100 \). The joint cdf \( F_X \) is defined as in (18) with the AMH copula. In Fig. 1, we depict the exact values of \( F_\alpha \) obtained with Algorithm 6 (\( \alpha = 0.5 \) and 0.9). For comparison purposes, the exact values of \( F_{\Sigma} \), where \( \Sigma \) is the sum of the independent rvs \( X_{11}, \ldots, X_{100} \) and \( X_i \sim X_i \) for \( i = 1, 2, \ldots, 100 \), are also provided. It is well known that the AMH copula introduces a low to moderate positive dependence relation between the rvs \( X_{i1}, \ldots, X_{100} \). However, the impact is clearly significant when the number of risks of the portfolio becomes large as illustrated in Fig. 1. In Table 3, we provide the values of \( E[S], \text{Var}(S), \text{VaR}(S) \) and \( \text{TVaR}(S) \). Note that the computation time increases as the dependence parameter becomes larger. Indeed, \( \theta^* = F_{\theta_0}^{-1}(1-\epsilon; \alpha) \) increases with \( \alpha \). For example, if \( \epsilon = 10^{-10} \), \( \theta^* = 34 \) and \( \theta^* = 219 \) for \( \alpha = 0.5 \) and \( \alpha = 0.9 \) respectively.

Example 8 allows us to better understand the impact of the dependence between individual risks on the overall exposure evaluation. As shown in Fig. 1, using a portfolio of 100 risks highlights this significant impact. Indeed, one cannot neglect the dependence, even moderate, for a portfolio of large dimension. Note that the proposed algorithm allowed us to make such a conclusion, which is not possible using the convolution method in (15), since it only applies to small portfolios.

2.4. Discrete mixing \( \text{rv} \Theta \) and continuous marginals

Let us assume that \( \bar{X} = (X_1, \ldots, X_d) \) is a vector of continuous positive rvs where the multivariate cdf is defined by an Archimedean copula \( C \) as in (5) and the mixing \( \text{rv} \Theta \) is discrete. In this case, the computation of \( F_{\Sigma\mid\theta_0=\theta} \) or eventually \( E[\bar{X} \mid \Theta = \theta] \) becomes more difficult. Since \( (X_i | \Theta = \theta) \) are conditionally independent rvs, one can apply the tools at hand to compute an accurate (as possible) approximation of \( F_{\Sigma\mid\theta_0=\theta} \) or \( E[\bar{X} \mid \Theta = \theta] \) for the sum or any function of independent rvs. Note that the approach described here can be easily adapted when the multivariate survival function instead of the multivariate cdf is defined with an Archimedean copula \( C \) as in (8) and when the mixing \( \text{rv} \Theta \) is discrete.

Inspired by Bargès et al. (2009), we propose an approximation based on discretization methods and the application of Algorithm 6. This approximation leads to numerical bounds as accurate as one may desire. Indeed, we approximate \( \bar{X} \) by \( \bar{X} = (\tilde{X}_1, \ldots, \tilde{X}_d) \), a vector of discrete rvs with \( \tilde{X}_i \in A = [0, 1h, 2h, \ldots] \) for \( i = 1, \ldots, d \) and a discretization step \( h > 0 \). The multivariate cdf of \( \bar{X} \) is defined with the same copula \( C \).
and with marginals $F_{X_1}, \ldots, F_{X_d}$ obtained with a discretization method, i.e., $F_{X_i}(k_i h, \ldots, k_d h) = C(F_{X_1}(k_1 h), \ldots, F_{X_d}(k_d h))$, for $k_1, \ldots, k_d \in \mathbb{N}_0$. In this article, we consider the upper, lower and mean preserving discretization methods (see e.g. Müller and Stoyan, 2002 or Bargès et al., 2009 for details).

Example 9. Let $X_i \sim \text{Exp}(0.1)$, for $i = 1, 2, \ldots, d$ $(d = 40)$. The multivariate cdf of $X$, $F_X$, is defined as in (5) with the AMH copula $(\alpha = 0.5)$. For $X = (X_1 + \cdots + X_d)/(d)$, we compute the upper and lower bounds, $F_U$ with $h = 1$ and $h = 0.1$. Values of the expectation, variance, and mean preserving discretization methods. Also, the approximated values of $E[X], \text{Var}(X), \text{TVaR}_0(X)$ and $\text{TVaR}_0(X)$ obtained using 100 000 MC simulations are provided (with 95%-level confidence intervals given in the last column). Clearly, as the discretization step $h$ goes to 0, the difference between the bounds also tends to 0. The proposed approximation has the advantage to allow us to control the precision of the approximation. The left panel of Fig. 2, the exact values of $F_{S_i}$, where $S$ is the sum of the independent rvs $X_1, \ldots, X_{100}$ and $X_i \sim \text{Exp}(0.1)$, are also depicted.

Table 4

<table>
<thead>
<tr>
<th>$S_{&lt;0.1}$</th>
<th>$S_{=0.1}$</th>
<th>$S_{&gt;0.1}$</th>
<th>$S_{m,0.1}$</th>
<th>$S_{m,0.1}$</th>
<th>MC</th>
<th>95%-confidence interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>380.3333</td>
<td>398.0033</td>
<td>402.0033</td>
<td>402.0033</td>
<td>400</td>
<td>400</td>
<td>399.8067 [398.8303; 400.7831]</td>
</tr>
<tr>
<td>2467.6236</td>
<td>2474.6556</td>
<td>2474.6556</td>
<td>2474.6556</td>
<td>2479.9950</td>
<td>2479.9523</td>
<td>2481.9510 [2460.8436; 2506.9318]</td>
</tr>
<tr>
<td>606</td>
<td>624</td>
<td>628.4</td>
<td>646</td>
<td>627</td>
<td>626.4</td>
<td>627.3094 [624.8721; 629.6699]</td>
</tr>
<tr>
<td>975</td>
<td>993.2</td>
<td>997.2</td>
<td>1015</td>
<td>995</td>
<td>995.2</td>
<td>989.2484 [981.2374; 1000.4592]</td>
</tr>
<tr>
<td>705.6048</td>
<td>723.3208</td>
<td>727.3191</td>
<td>745.6048</td>
<td>725.7719</td>
<td>725.3294</td>
<td>726.2473 [725.7238; 726.7709]</td>
</tr>
<tr>
<td>1027.2863</td>
<td>1043.7218</td>
<td>1047.7156</td>
<td>1067.2863</td>
<td>1047.4537</td>
<td>1047.7465</td>
<td>1039.4606 [1039.1150; 1039.8062]</td>
</tr>
</tbody>
</table>

Fig. 2. Approximated values of the cdf of $S = X_1 + \cdots + X_{100}$.

2.5. Continuous mixing rv $\theta$

The common mixture representation in (6) and (9) leads to a two-step natural sampling procedure for $X$. The first step is to simulate a sampled value of $\theta$. Then, in the second step, the sampled value of $X$ is computed via the conditional distribution of $X$ given the sampled value of $\theta$ using either (7) or (10). See, e.g., Marshall and Olkin (1988) or Hoeffnig (2008) for details. We examine two alternatives to this approach: one based on the simulation of the mixing rv $\theta$ and another one on the approximation of $\theta$ by a discrete rv.

We consider a vector of discrete (or discretized) rvs $X = (X_1, \ldots, X_d)$ where $X_i \in \{0, 1h, 2h, \ldots\}$ $(i = 1, \ldots, d)$. Let the rv $Z$ be the estimator of $E[X_1, \ldots, X_d]$ under a MC simulation approach, which is constructed using the two-step sampling procedure of $X$. The standard error of the estimator $Z$ is given by $\sqrt{\text{Var}(Z)}$. We can use a variance reduction technique, namely the conditional MC simulation method (see, e.g., Lemieux, 2009 or Kroese et al., 2013 for details on this topic) to reduce this standard error. First, one generates sampled values of the rv $\theta$ and then applies Algorithm 6 as follows. We produce $m$ sampled values of $\theta$, denoted by $\theta(1), \ldots, \theta(m)$. Then, (27) becomes $f_z (kh) \simeq \sum_{\theta} f(z|h,\theta) (kh) \frac{1}{\theta}, k \in \mathbb{N}_0$, where the values of $f(z|h,\theta)$ are computed using e.g. FFT or DePril’s Algorithm. Then, $E[Z|\theta]$ corresponds to the resulting conditional MC approximation of $\psi(X_1, \ldots, X_d)$. Clearly, the standard error of $E[Z|\theta]$ is given by $\sqrt{\text{Var}(Z|\theta)}$. Since

$$\text{Var}(Z) = E[\text{Var}(Z|\theta)] + \text{Var}(E[Z|\theta]),$$

it is clear that

$$\sqrt{\text{Var}(E[Z|\theta])} \leq \sqrt{\text{Var}(Z)}.$$ (31)

The magnitude of the difference $\sqrt{\text{Var}(Z)} - \sqrt{\text{Var}(E[Z|\theta])}$ is analyzed in the following example.

Example 10. Let $X^{(C,s,a)} = (X_1^{(C,s,a)}, \ldots, X_{50}^{(C,s,a)})$ and $X^{(G,s,a)} = (X_1^{(G,s,a)}, \ldots, X_{50}^{(G,s,a)})$ be two vectors of rvs with $X_i^{(C,s,a)} \sim \text{Bin}(100, 0.2)$, for $i = 1, \ldots, 50$, where their joint survival functions are defined either with the Clayton copula.
or the Gumbel copula according to (8). Similarly, let \( \text{Var}(S_{G,f}) \) and \( \text{Var}(S_{C,f}) \) be two vectors of rv's with \( X_{1,f}^{(i)} \sim X_{1,f}^{(i)} \sim \text{Bin}(100, 0.2) \), for \( i = 1, \ldots, 50 \), where their joint cdfs are defined either with the Clayton copula or the Gumbel copula according to (5). For each copula, the dependence parameter \( \alpha \) of the copula is fixed such that Kendall’s tau is equal to 0.2, 0.5, or 0.8. The exact values of the covariances between pairs of rv's for both vectors of dependency parameter are provided in Tables 5 and 6 for the three values of the dependence parameter. We define \( S_{G,f} = \sum_{i=1}^{50} Y_{1,f}^{(i)}, S_{C,f} = \sum_{i=1}^{50} Y_{1,f}^{(i)} \), and \( S_{C,f} = \sum_{i=1}^{50} Y_{1,f}^{(i)} \). Clearly, \( E[S_{G,f}] = E[S_{C,f}] = E[S_{C,f}] = 1000 \), whatever the dependence parameter.

In Tables 7 and 8, we also give the exact values of \( \text{Var}(S_{G,f}), \text{Var}(S_{C,f}) \), \( \text{Var}(S_{C,f}) \), \( \text{Var}(S_{G,f}) \), and \( \text{Var}(S_{C,f}) \) for the three values of their dependence parameter. In Tables 9–10, we provide the approximated values of \( \overline{F}_{G,f}(xi), \overline{F}_{G,f}(xi), \overline{F}_{G,f}(xi), \overline{F}_{G,f}(xi), \overline{F}_{G,f}(xi) \). Those values are computed using both the conditional MC and the MC approaches with \( m = 100000 \) simulations. In parenthesis, we indicate the values of the standard deviation for each approximation. As expected from (31), we observe that the standard error of the approximation based on the conditional MC approach is lower than the corresponding one for the approximation based on the MC approach. For a given multivariate distribution, we observe that the improvement is more significant as the dependence parameter \( \alpha \) decreases. The improvement is also more significant for large values of \( \alpha \). However, for a specific value of Kendall’s tau, the improvement differs from one multivariate distribution to another. Notably, we observe that the improvement is the least significant for the results associated to \( X_{1,f}^{(i)} \) and \( X_{1,f}^{(i)} \), meaning that the improvement observed with the conditional MC approach is less significant with multivariate distributions having a non-zero right tail dependence.

Let us now consider the case of the Clayton copula for which \( \Theta = \text{gamma} \) distributed. To obtain \( f_{\Theta} \), we can approximate the Clayton copula with the shifted negative binomial copula allowing us to use Algorithm 6. This family includes the AMH copula as a special case and the Clayton copula as a limit case (see Cossette et al., 2017 for more details). The idea here is to approximate the rv \( \Theta \) by a discrete rv \( \Theta \) and apply the proposed methodology of Section 2.3. However, we need to be careful. We cannot blindly apply the three discretization methods used in the previous section because we aim to approximate the copula generated from \( \Theta \) and not only the distribution of \( \Theta \). Thus, finding an appropriate discrete rv \( \Theta \) is not an easy task as we will see in the special case of the Clayton copula.

The multivariate shifted negative binomial copula is defined by

\[
E_{\Theta}(\alpha)_{CMCapprox.} = \left( q_{h} \left( \frac{d}{\Theta} (q_{h}u_{j}^{(n)}(1 - q_{h})) - (1 - q_{h}) \right) \right)^{-1/\alpha}.
\]

The two parameters of the copula are \( \alpha > 0 \) and \( q_{h} = 1 - e^{-h} \), where \( h > 0 \) can be seen as a discretization parameter.

The underlying mixing rv, associated to (32) and denoted by \( \Theta_{CMCapprox.}^{SNB}(\alpha) \), follows a shifted negative binomial distribution, i.e., the rv \( \Theta_{CMCapprox.}^{SNB}(\alpha) \) is defined as \( \Theta_{CMCapprox.}^{SNB}(\alpha) = h (M_{\Theta}(\alpha) + \alpha) \) where \( M_{\Theta}(\alpha) \) follows a negative binomial distribution, i.e., \( M_{\Theta}(\alpha) \sim NB(\frac{1}{\alpha}, q_{h}) \), with \( f_{\Theta}(\alpha) = \left( \frac{1}{\alpha} \right)^{k} \left( 1 - q_{h} \right)^{k} \), \( k \in \mathbb{N}_0 \), and \( E \left[ M_{\Theta}(\alpha) \right] = 1 - \frac{1}{\alpha} \) with \( q_{h} = 1 - e^{-h} \), \( h > 0 \). The LST of \( \Theta_{CMCapprox.}^{SNB}(\alpha) \) is

\[
L_{\Theta_{CMCapprox.}^{SNB}(\alpha)}(t) = \left( e^{-th} - e^{-(1-t)h} + \frac{1}{1-e^{-h}} \right)^{1/\alpha}.
\]
Indeed, (32) can be represented as

\[ e_{\alpha,0}^{\text{SNB}}(t_1, \ldots , t_d) = \mathcal{L}_{\alpha,0}(n) \left( \mathcal{L}_{\alpha,0}^{-1}(n)(t_1) + \cdots + \mathcal{L}_{\alpha,0}^{-1}(n)(t_d) \right). \]

When \( d = 2, q_0 = 1 - \beta \) and \( \gamma = \frac{1}{2}, (32) \) becomes

\[ e_{\alpha,0}^{\text{SNB}}(t_1, t_2) = \left( 1 - \beta \right) \left( \sum_{i=1}^{2} \left( 1 - \beta ight) u_i^{1/2} + \beta \right) \gamma, \]

which corresponds to the so-called bivariate Lomax copula in Balakrishnan and Lai (2009), bivariate Fang–Fang–Rosen copula in Fang et al. (2000) and Genest and Rivest (2001), and bivariate BB10 copula in Joe (2014). Note that the multivariate copula provided in (2.2) of Fang et al. (2000) does not correspond to the Archimedean copula in (32). The copula in (32) is constructed via the approach proposed by Marshall and Olkin (1988) as for the BB10 copula.

Clearly, when \( h \to 0 \), we have

\[ \lim_{h \to 0} \left[ e_{\alpha,0}^{\text{SNB}}(t_1, \ldots , t_d) \right] = \left( \frac{1}{1 + t} \right)^{\frac{1}{2}}, \]

where \( \left( \frac{1}{1 + t} \right)^{\frac{1}{2}} \) is the LST of the rv \( \Theta \sim \text{Gamma} \left( \frac{1}{a}, 1 \right) \). Then, given

\[ \Theta \sim \text{Gamma} \left( \frac{1}{a}, 1 \right), \]

we have

\[ \Theta^{(j, k)} \sim \text{Gamma} \left( \frac{1}{a}, 1 \right). \]

The shifted negative binomial copula is an Archimedean copula. Indeed, (32) can be represented as

\[ e_{\alpha,0}^{\text{SNB}}(t_1, \ldots , t_d) = \mathcal{L}_{\alpha}(h) \left( \mathcal{L}_{\alpha}^{-1}(h)(t_1) + \cdots + \mathcal{L}_{\alpha}^{-1}(h)(t_d) \right). \]

As a special case, when \( \alpha = 1 \), the shifted negative binomial copula in (32) becomes the AMH copula. For a fixed \( \alpha > 0 \), when \( h \to 0 \) (i.e., \( q_0 \to 1 \)), the limit of the shifted negative binomial copula corresponds to the Clayton copula with parameter \( \alpha \), i.e.,

\[ \lim_{h \to 0} e_{\alpha,0}^{\text{SNB}}(t_1, \ldots , t_d) = \left( \sum_{i=1}^{d} t_i \right)^{-\frac{1}{\alpha}}. \]

In the following example, we examine the efficiency of the approximation of the Clayton copula with the shifted negative binomial copula in a risk aggregation context.

**Example 11.** In this example, we compare the performance of the approximation of the Clayton copula by the shifted negative binomial copula. Let \( X_i^{(\alpha)} = \left( X_i^{(\alpha)}(1), \ldots , X_i^{(\alpha)}(d) \right) \) and \( X_i^{\text{SNB}(a,h)} = \left( X_i^{\text{SNB}(a,h)}, \ldots , X_i^{\text{SNB}(a,h)} \right) \) be two vectors of rvs with \( X_i^{(\alpha)} \sim X_i^{(\alpha)} \sim \text{Bin} \left( 10, 0.2 \right) \), for \( i = 1, \ldots , 4 \), where their joint cdfs are defined either with the Clayton copula (with \( \alpha = 0.5, 2, 8 \)) or the shifted negative binomial copula (with \( \alpha = 0.5, 2, 8 \) and \( h = 0.001, 0.0001 \)) according to (5). We define \( X_i^{(\alpha)} = \sum_{j=1}^{d} X_i^{(\alpha)}(j) \) and \( S^{(\text{SNB}(a,h))} = \sum_{j=1}^{d} X_i^{(\text{SNB}(a,h))(j)} \). Both the exact values of \( F_i^{(\alpha)} \) resulting from (15) and \( F_i^{(\text{SNB}(a,h))} \) using the shifted negative binomial copula are given in Tables 15–17. The results of both the conditional and the full MC simulation methods are also provided. The approximation using the shifted negative binomial copula is good and more significant as \( h \) decreases and for Kendall tau
below 0.5. The stronger the dependence relationship, the more the calculation time increases. Once again, the efficiency of the approximation of a copula with non-zero lower tail dependence by another one with \( \lambda_i = 0 \) is less significant when the related dependence relationship is strong.

To summarize, if the mixing \( \text{rv} \Theta \) is continuous, we can proceed either by conditional MC simulation or by the approximation of the \( \text{rv} \Theta \) with a discrete one. The difference between both approximation methods depends on several elements such as the value of the chosen dependence parameter, the size of the portfolio, the number of simulations, etc. For example, the approximation of the Clayton copula with the shifted negative binomial copula is more accurate compared to the simulation method when the dependence parameter is small. In the latter case, execution times of both methods are comparable. However, in the event of a greater degree of dependence or a greater number of risks, the conditional MC simulation executes faster. Note that the aim here was to only present different approaches to solve this problem, rather than putting them through a comprehensive comparison.

### 2.6. Portfolio of exchangeable risks

Exchangeability plays an important role in the analysis of homogeneous portfolios in actuarial science and in quantitative risk management, notably in credit risk modeling (e.g. McNeil et al., 2015 and the references therein). Based on De Finetti’s Theorem (see De Finetti, 1957) and its extension in Bühmann (1960), an infinite exchangeable sequence of rv can be represented as a mixture over a common parameter \( \Theta \) of an infinite sequence of iid rv (see e.g. Felder, 1971 for more details). Let \( X = (X_n; n \in \mathbb{N}) \) be a sequence of positive and exchangeable rv. Define the sequence \( W = (W_n; n \in \mathbb{N}) \), where \( W_n = \frac{X_1 + \cdots + X_n}{n} \) for \( n \in \mathbb{N} \). As discussed in, e.g. Aldous (1985), the common parameter \( \text{rv} \) drives the asymptotic behavior of \( W \). In the context of credit risk, Frey and McNeil (2001, 2002) have studied the asymptotic behavior of \( W \) with the distribution of \( (X_1, \ldots, X_N) \) defined in terms of exchangeable Bernoulli mixture models, for \( n = 2, 3, \ldots \) (see e.g. Proposition 3.2 in Frey and McNeil, 2002). Their result can be generalized to other dependence models. In the context of Section 2.3, we consider the joint distribution of \( (X_1, \ldots, X_N) \) defined with an Archimedean copula \( C \) with either (18) or (19) for \( n = 2, 3, \ldots \). Let \( Z \) be a discrete \( \text{rv} \) with \( Z \in \{ z_0, \theta \in \mathbb{N} \} \) where \( z_0 = E[X|\Theta = \theta] < \infty \) and \( f_\theta (z_0) = f_\theta (\theta) \), \( \theta \in \mathbb{N} \). Clearly, the sequence \( W_n \) converges in distribution to the \( \text{rv} \) \( Z \), i.e.,

\[
W_n \xrightarrow{D} Z. \tag{34}
\]

Indeed, we have

\[
\mathcal{L}_{W_n} (t) = \sum_{\theta=1}^{\infty} \mathcal{L}_{W_{n|\Theta = \theta}} (t) f_\theta (\theta) / \mathcal{L}_{X|\Theta = \theta} (z_0 n) = \sum_{\theta=1}^{\infty} \left( \frac{\mathcal{L}_{X|\Theta = \theta} (z_0)}{n} \right) f_\theta (\theta). \tag{34}
\]

Near the origin, \( \mathcal{L}_{W_{n|\Theta = \theta}} (t) = 1 - z_0 t + o (t) \) which implies that

\[
\lim_{n \to \infty} \mathcal{L}_{W_n} (t) = \sum_{\theta=1}^{\infty} \left( f_\theta (\theta) \lim_{n \to \infty} \left( 1 - z_0 t / n \right)^n \right) = \sum_{\theta=1}^{\infty} f_\theta (\theta) e^{-z_0 t} = \mathcal{L}_Z (t), \tag{34}
\]

leading to (34).

The result in (34) shows that for a large portfolio of exchangeable risks, the distribution of \( W_n \) can be approximated by the distribution of \( Z \). One of the interesting features of the strategy described in Section 2.3 is that the exact values of \( z_0 (\theta \in \mathbb{N}) \) can be computed as well as its pmf. If the \( \text{rv} X \) is continuous, discretization methods presented in Section 2.4 are used.

Let us look at a simple example that illustrates the convergence of \( W_n \) to \( Z \) for a portfolio of 100 exchangeable risks.
Example 12. Let \( X_1, \ldots, X_{100} \) be a vector of 100 exchangeable rvs with \( X_i \sim X \sim \text{Pois} (\lambda = 5) \), for \( i = 1, \ldots, 100 \). The multivariate cdf of \( X_1, \ldots, X_{100} \) is defined with an AMH copula with dependence parameter \( \alpha = 0.5 \) and \( \alpha = 0.9 \).

As shown in Table 18 and illustrated in Fig. 3, for large portfolios, the distribution of \( Z \) is a good candidate to approximate the behavior of \( W_n \). This result is very important in terms of computation time for very large portfolios since using \( Z \) is a faster, more efficient tool in comparison to directly computing \( W_n \).

### 3. Capital allocation

Capital allocation is fundamental in actuarial science and quantitative risk management. It describes how the capital needed for the whole portfolio can be divided and allocated between risks of the portfolio. It is crucial for an insurance company or a financial institution to evaluate the overall capital charge for a portfolio of risks in order to protect itself from large rare events. The amount of capital needed for the entire portfolio is determined with a chosen risk measure \( \rho \). Among the desired properties for a capital allocation rule (see e.g. Furman and Zitikis, 2008), one has to be fully additive, i.e.,

\[
\rho (S) = \sum_{i=1}^{n} C_i, \tag{35}
\]

where \( S = X_1 + \cdots + X_n \) and \( C_i \) is the contribution of the \( i \)-th risk to the aggregate risk of the portfolio.

In the present section, we show how to compute the values (exact or approximated) of the contributions \( C_i \) with the proposed methodology of Section 2. We consider Euler’s capital allocation principle and the weighted risk capital allocation principle using different risk measures.

Let us assume here that we are in the context of Section 2.3, meaning that \( \Theta \) is a discrete rv defined on \( \mathbb{N} \) and that \( X_i \in A = \{0, 1h, 2h, \ldots \} \) (\( i = 1, \ldots, n \)). Also, assume that the joint cdf of \( X \) or its joint survival function is defined with the Archimedean copula \( C \) as in (18) or (19). If the rvs \( X_i \) are continuous, the procedure described in Section 2.4 will be used to find the contributions. For a continuous \( \Theta \), we refer to Section 2.5.

To apply Euler’s capital allocation rule, we need to assume that the risk measure \( \rho \) is positive homogeneous. Let us define

\[
L (\lambda) = \sum_{i=1}^{n} \lambda_i X_i,
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_n) \). For a given risk measure \( \rho \), the contribution \( C_i \) allocated to risk \( i \) is given by

\[
C_i = \rho (L; X_i) = \lambda_i \frac{\partial}{\partial \lambda_i} \rho (L (\lambda)) \bigg|_{\lambda = 1}^\lambda, \text{ for } i = 1, 2, \ldots, n,
\]

where \( \lambda = (1, \ldots, 1) \). We apply the Euler allocation principle with three different risk measures: covariance, VaR and TVaR (see Table 19 for their expressions). For more details, see e.g. Tasche (1999), Embrechts et al. (2005) or Rosen et al. (2011).

The second capital allocation principle we use is the one proposed by Furman and Zitikis (2008) in which the capital amount required and the contributions are based on weighted risk measures. Furman and Zitikis (2008) propose to compute the capital amount with weighted risk measures and to use weighted allocation methods for determining the contribution of each risk \( i, i = 1, 2, \ldots, n \). Let \( \omega \) be a weight function. Then, the capital amount corresponds to \( \rho_\omega \), which is defined by \( \rho_\omega (S) = E[\Theta \omega (S)] / E[\Theta] \), assuming that both expectations exist. Since

\[
\frac{E[\Theta \omega (S)]}{E[\Theta]} = E \left[ \frac{\sum_{i=1}^{n} \omega(X_i)}{E[\Theta]} \right] = \sum_{i=1}^{n} \frac{E[X_i \omega]}{E[\Theta]}, \tag{36}
\]

the share of the capital allocated to \( X_i \) is given by

\[
C_i = \rho_\omega (X_i; S) = \frac{E[X_i \omega (S)]}{E[\omega (S)]},
\]

for \( i = 1, 2, \ldots, n \). Clearly, due to (36), property (35) is satisfied.

### Table 18

<table>
<thead>
<tr>
<th>( \alpha = 0.5 )</th>
<th>( \alpha = 0.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E (W_n) )</td>
<td>5</td>
</tr>
<tr>
<td>( \text{Var} (W_n) )</td>
<td>0.920999</td>
</tr>
<tr>
<td>( \text{VaR}_{0.9} (W_n) )</td>
<td>6.39</td>
</tr>
<tr>
<td>( \text{TVaR}_{0.9} (W_n) )</td>
<td>7.34</td>
</tr>
<tr>
<td>( \text{VaR}_{0.9999} (W_n) )</td>
<td>8.31</td>
</tr>
<tr>
<td>( \text{TVaR}_{0.9999} (W_n) )</td>
<td>8.383850</td>
</tr>
<tr>
<td>( \text{TVaR}_{0.9999} (W_n) )</td>
<td>7.599997</td>
</tr>
<tr>
<td>( \text{TVaR}_{0.9999} (W_n) )</td>
<td>8.448253</td>
</tr>
</tbody>
</table>

\( n = 5 \) and \( \alpha = 0.5 \) of the contributions \( C_i \) where \( \rho \) is chosen risk measure of capital needed for the entire portfolio is determined with a chosen risk measure \( \rho \).
Among the several weight functions that can be used to calculate the capital allocation, we consider the following three methods: Esscher with $\alpha(s) = e^s$, Kamp with $\alpha(s) = 1 - e^{-as}$ and size-biased with $\alpha(s) = s^\alpha$.

Let us address the computation of $C_i$ for the considered Euler capital allocation rules and weighted risk capital allocation rules given in Table 19. One encounters frequently the evaluation of the expectation $E[X; S]$ (or a variation of it) which poses problem given that $X_i$ and $S$ are dependent. To circumvent this, we rewrite the product of these two rvs as

$$E[X; S] = E_0 \left[ E \left[ X_i (S + S_{-i}) | \theta \right] \right]$$

where $S_{-i} = X_1 + \cdots + X_{i-1} + X_{i+1} + \cdots + X_n$. The conditional independence of $X_i$ and $S_{-i}$ given $\theta$ simplifies the evaluation of the quantities of interest.

Note that with the VaR risk measure, it is slightly more tedious. In this case, the contribution of risk $i$ is given by

$$C_i = E \left[ X_i | S = \text{VaR}_\theta (S) \right] = E \left[ X_i | S = k_0 h \right] = \frac{E \left[ X_i \times 1_{S=k_0 h} \right]}{\text{Pr} \left( S = k_0 h \right)},$$

assuming that $\text{Pr} \left( S = k_0 h \right) > 0$. Then, using our approach, we have

$$E \left[ X_i \times 1_{S=k_0 h} \right] = E_\theta \left[ E \left[ X_i \times 1_{S=k_0 h} | \theta \right] \right]$$

$$= \sum_{\theta=1}^{\infty} \left( \sum_{j=1}^{k_0} f_{X_i|\theta=\theta} (j) f_{S_{-i}|\theta=\theta} (k_0 - j) h \right) f_\theta (\theta)$$

$$= \sum_{\theta=1}^{\infty} E \left[ X_i | \theta = \theta \right] \left( \sum_{j=1}^{k_0} f_{X_i|\theta=\theta} (j) \right) f_{S_{-i}|\theta=\theta} (k_0 - j) h \right) f_\theta (\theta),$$

with $f_{X_i|\theta=\theta} (j) = \frac{g_{X_i|\theta=\theta} (j)}{f_{S_{-i}|\theta=\theta} (k_0 - j) h}$, for $j \in \mathbb{N}$. Finally, defining

$$g_{X_i|\theta=\theta} (k_0 h) = \sum_{j=1}^{k_0} f_{X_i|\theta=\theta} (j) f_{S_{-i}|\theta=\theta} (k_0 - j) h$$

as a result of a convolution product, we obtain

$$E \left[ X_i \times 1_{S=k_0 h} \right] = \sum_{\theta=1}^{\infty} E \left[ X_i | \theta = \theta \right] g_{X_i|\theta=\theta} (k_0 h) f_\theta (\theta),$$

(37)

where the values of $g_{X_i|\theta=\theta} (k_0 h)$ can be easily obtained using classic aggregation methods such as FFT.

**Example 13.** We consider a portfolio of 10 risks, where $X_i - 1 \sim \text{NB} \left( \frac{1}{2}, q_i \right), \text{ such that } E[X_i] = 5, \text{ for } i = 1, 2, \ldots, 10$. It implies $E[S] = 50$. The multivariate cdf of $X$ is defined as in (18) with the AMH copula. For $\alpha = 0, 0.3$ and 0.8, the variance of $S$ is 104.6361, 185.2821 and 343.7994 respectively. In Table 20, we provide the relative contributions of $X_1, \ldots, X_{10}$ (i.e., $\frac{C_i}{\sum_{i=1}^{10} C_i}$) for the methods based on Euler’s capital allocation rule and the ones under the weighted risk allocation approach assuming a dependence parameter $\alpha = 0.8$. As shown in Table 20, we are able to find exact contributions based on different allocation methods. Consistent results are obtained for $\alpha = 0$ and $\alpha = 0.3$.

### 4. Random sum of exchangeable risks

Random sums are essential in the description of many fundamental risk models in actuarial science. The so-called frequency-severity model is based on random sums. In this section, the computational methodology exposed in Section 2.3 is used to analyze the distribution of the aggregate claim amount $S$ which is defined as the random sum of exchangeable individual claim amounts. Under this risk model, the rv $S$ is defined by $S = \sum_{i=1}^{N} X_i$, where $X_i = \{X_i, j \in \mathbb{N}\}$ forms a sequence of exchangeable rvs independent of the counting positive discrete rv $N$. The components of $X$ are such that $X_i \in \mathcal{A} = \{0, 1h, 2h, \ldots \} (j \in \mathbb{N})$. The notation for the univariate pmf of $X_i$ and the univariate cdf of $X_i$ is given in Section 2. Also, we assume that the joint distribution of $(X_1, \ldots, X_N)$ is defined via either (18) or (19) with $d = 1$.

The risk model considered in this section can be seen as an extension of the risk model described and studied in Section 2 of Albrecher et al. (2011). Indeed, they consider a risk model defined with a compound Poisson process where the vector of claim amounts follows a multivariate mixed exponential distribution as defined in Section 2.2. Here, the multivariate distribution of the vector of claim amounts can be defined with any Archimedean copula and any marginal distributions for the claim amounts.

To apply the computational methodology, assume that the mixing rv $\theta$ is a strictly positive discrete rv defined on $\mathbb{N}$. Clearly, the aggregate claim amount $S \in \mathcal{A}$ and the objective is to compute the values of $f_S (k h) = \text{Pr} (S = k h), k \in \mathbb{N}_0$. Given the common mixture representation of either $F_{X_1, \ldots, X_N}$ or $F_{X_1, \ldots, X_N}$, we have $f_S (k h) = \sum_{\theta=1}^{\infty} f_{S|\theta=\theta} (k h) f_\theta (\theta)$, where $(S|\theta = \theta) = \sum_{\theta=1}^{\infty} (X_i|\theta = \theta)$. The usual aggregation algorithms (see e.g. Panjer et al., 2008) such as FFT or Panjer’s recursive algorithm can be used to compute the values of $f_{S|\theta=\theta} (k h)$ for $k \in \mathbb{N}_0$ and for each $\theta = 1, 2, \ldots, \theta^*$ where $\theta^*$ is chosen such that $F_{S|\theta^*} (\theta^*) \approx 1 - \epsilon$ where $\epsilon$ is fixed as small as desired (e.g. $\epsilon = 10^{-10}$). Again, note that we just need the values of $f_{X_i|\theta=\theta} (k h)$, which are given in (22) or (25).

Also, the LST of the rv $S$ is given by

$$\mathcal{L}_S \left( t \right) = E \left[ e^{-t S} \right] = \sum_{\theta=1}^{\infty} E \left[ e^{-t S|\theta=\theta} \right] f_\theta (\theta)$$

$$= \sum_{\theta=1}^{\infty} \mathcal{L}_{S|\theta=\theta} \left( t \right) f_\theta (\theta),$$

where...
Table 20
Contributions of $X_j$ ($j = 1, 2, \ldots, 10$) in % under the 6 allocation methods. The multivariate cdf of $X$ is defined with an AMH copula ($\alpha = 0.8$).

<table>
<thead>
<tr>
<th>$i$</th>
<th>Covariance</th>
<th>VarR</th>
<th>TVaR</th>
<th>Esscher ((\psi = 1))</th>
<th>Kamps ((\psi = 1))</th>
<th>Size – biased ((\psi = 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11.53</td>
<td>28.52</td>
<td>31.92</td>
<td>14.23</td>
<td>10.55</td>
<td>14.35</td>
</tr>
<tr>
<td>2</td>
<td>10.75</td>
<td>11.88</td>
<td>11.65</td>
<td>11.52</td>
<td>10.27</td>
<td>11.58</td>
</tr>
<tr>
<td>3</td>
<td>10.32</td>
<td>9.17</td>
<td>8.80</td>
<td>10.45</td>
<td>10.12</td>
<td>10.48</td>
</tr>
<tr>
<td>4</td>
<td>10.04</td>
<td>8.15</td>
<td>7.75</td>
<td>9.85</td>
<td>10.01</td>
<td>9.87</td>
</tr>
<tr>
<td>5</td>
<td>9.84</td>
<td>7.61</td>
<td>7.21</td>
<td>9.47</td>
<td>9.94</td>
<td>9.47</td>
</tr>
<tr>
<td>6</td>
<td>9.69</td>
<td>7.28</td>
<td>6.88</td>
<td>9.21</td>
<td>9.89</td>
<td>9.20</td>
</tr>
<tr>
<td>7</td>
<td>9.58</td>
<td>7.05</td>
<td>6.65</td>
<td>9.01</td>
<td>9.85</td>
<td>8.99</td>
</tr>
<tr>
<td>8</td>
<td>9.49</td>
<td>6.89</td>
<td>6.49</td>
<td>8.86</td>
<td>9.81</td>
<td>8.84</td>
</tr>
<tr>
<td>9</td>
<td>9.41</td>
<td>6.77</td>
<td>6.37</td>
<td>8.75</td>
<td>9.79</td>
<td>8.71</td>
</tr>
<tr>
<td>10</td>
<td>9.35</td>
<td>6.68</td>
<td>6.28</td>
<td>8.65</td>
<td>9.77</td>
<td>8.61</td>
</tr>
</tbody>
</table>

The claim amount rvs $X = \{X_j, j \in N\}$, where $X_j$ corresponds to the amount of the $j$th claim, are assumed to be a sequence of strictly positive and iid rvs with pdf $f_X$ and cdf $F_X$. The sequences $W$ and $X$ are independent.

The time of ruin is defined by the rv $\tau_u = \inf\{t \geq 0 : U(t) < 0\}$, with $\tau_u = \infty$ if $U(t) \geq 0$ for all $t \geq 0$. The infinite-time ruin probability is $\zeta(u) = \Pr(\tau_u = \infty | U(0) = u)$. Throughout this section, we assume the positive security loading condition $E[cW - X] > 0$ to be verified which ensures that ruin will not occur almost surely. Due to the common mixture representation, (38) is given by

$$F_{W|\theta\in\Theta}(x) = F_W(x) \cdot \frac{\Theta}{\Theta + 1} \cdot \frac{\Theta + 1}{\Theta + 2} \cdot \frac{\Theta + 2}{\Theta + 3} \cdot \ldots \cdot \frac{\Theta + k}{\Theta + k + 1}$$

for $x \geq 0$. As mentioned in Section 3 of Albrecher et al. (2011), $F_{W|\theta\in\Theta}$ is the canonical survival function of the inter-claim time rvs for an ordinary renewal process. Let $\zeta_\theta$ be the conditional ruin probability associated to the corresponding renewal process. It implies that the ruin probability $\zeta$ can be represented as a mixture, where $\phi$ is the mixing rv, i.e.,

$$\zeta(u) = \int_0^\infty \zeta_\theta(u) \cdot dF_\phi(\theta).$$

The security loading condition is violated when the mixing rv $\phi$ takes a value larger that $\theta_0 > 0$. We define $\theta_0$ such that $\zeta_\theta(u) = 1$, for $\theta > \theta_0 > 0$. There exists a $\theta_0$ such that $c \cdot E[W|\phi = \theta] > E[X]$, for $\theta \in \{1, 2, \ldots, \theta_0\}$, and $c \cdot E[W|\phi = \theta] < E[X]$, for $\theta \in \{\theta_0 + 1, \ldots\}$. Then, (39) becomes

$$\zeta(u) = \int_0^{\theta_0} \zeta_\theta(u) \cdot dF_\phi(\theta) + F_\phi(\theta_0),$$

for $u \geq 0$. Assuming that $\zeta_\theta$ could be computed for each $\theta \in \{1, 2, \ldots, \theta_0\}$ of the ordinary renewal process associated to $F_{W|\theta\in\Theta}$, the value of $\zeta(u)$ can be computed using (40).

In the examples of Section 3 of Albrecher et al. (2011), the authors assume that $(W_1, W_2, \ldots, W_k)$ follows a multivariate mixed exponential distribution as defined in Section 2.2, where the LST of the mixing rv corresponds to the generator of an Archimedean copula. It means that the univariate marginal distribution of the inter-claim time is a univariate mixed exponential distribution. Indeed, the authors consider specific examples of mixed Poisson risk models.

In this section, we show that it is possible to consider any multivariate distribution for $W$ defined with any Archimedean copula.
Table 21

<table>
<thead>
<tr>
<th></th>
<th>( \alpha = 0.8 ) (simul)</th>
<th>( \alpha = 0.8 ) (upper)</th>
<th>( \alpha = 0.8 ) (lower)</th>
<th>( \alpha = 0 ) (exact)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{E}[S] )</td>
<td>400.0359</td>
<td>399.9050</td>
<td>400.0500</td>
<td>400</td>
</tr>
<tr>
<td>( \bar{V} )</td>
<td>140.7755.887</td>
<td>140.912.018</td>
<td>140.942.018</td>
<td>80.000</td>
</tr>
<tr>
<td>( \bar{VaR}_\alpha )</td>
<td>906.9635</td>
<td>907.0000</td>
<td>907.2000</td>
<td>873.8748</td>
</tr>
<tr>
<td>( \bar{VaR}_0 )</td>
<td>1645.4705</td>
<td>1645.6000</td>
<td>1645.9008</td>
<td>1470.9808</td>
</tr>
<tr>
<td>( \bar{VaR}_{0.999} )</td>
<td>2317.5620</td>
<td>2323.300</td>
<td>2323.650</td>
<td>1992.0052</td>
</tr>
<tr>
<td>( \bar{VaR}_{0.999} )</td>
<td>2959.6858</td>
<td>2967.250</td>
<td>2967.600</td>
<td>2473.3833</td>
</tr>
<tr>
<td>( \bar{TVaR} )</td>
<td>1230.7370</td>
<td>1231.2106</td>
<td>1231.333</td>
<td>1138.1220</td>
</tr>
<tr>
<td>( \bar{TVaR} )</td>
<td>1939.1160</td>
<td>1941.278</td>
<td>1941.494</td>
<td>1699.2458</td>
</tr>
<tr>
<td>( \bar{TVaR}_{0.999} )</td>
<td>2596.5160</td>
<td>2603.6256</td>
<td>2603.9335</td>
<td>2202.1856</td>
</tr>
<tr>
<td>( \bar{TVaR}_{0.999} )</td>
<td>3223.8834</td>
<td>3236.6140</td>
<td>3237.032</td>
<td>2672.1090</td>
</tr>
</tbody>
</table>

Also, in order to calculate the adjustment coefficient \( \rho_0 \), we develop the expression of \( \mathbb{E} \left[ e^{-tW} | \Theta = \theta \right] \) as follows

\[
\mathbb{E} \left[ e^{-tW} | \Theta = \theta \right] = \int_0^\alpha e^{-tx} \frac{\theta qe^x}{(qe^x + 1 - q)^{\rho_0 + 1}}dx.
\]

Let \( u = qe^x + 1 - q \), then (45) becomes

\[
\int_1^\infty e^{-tx} \frac{\theta qe^x}{(qe^x + 1 - q)^{\rho_0 + 1}}dx = \int_1^\infty \frac{\theta q^x (u - q - 1)\,du}{(q - 1) t \left( \frac{u}{t+1} \right)^{\rho_0 + 1}}
\]

\[
= \frac{\theta q^x}{(q - 1) t} \times \left( \frac{q - 1}{q} \right) \frac{\Gamma(t, t + \theta)}{\Gamma(t + \theta + 1, 1 - q)}
\]

where \( \Gamma(c, d) \) is the generalized hypergeometric function defined as follows

\[
\Gamma(c, d) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_m)_k}{(b_1)_k \ldots (b_n)_k} \frac{z^k}{k!},
\]

with \( (x)_k = x(x + 1) \ldots (x + k - 1) \).

Let \( \lambda = 1, \beta = 1, \) and \( c = 1.2 \). The parameter of the AMH copula is \( \alpha = 1 - q = 0.9 \). We find that \( \theta_0 = 5 \) and the expression in (43) for \( \zeta(u) \) becomes

\[
\zeta(u) = \sum_{\theta=1}^{\theta_0} \Pr(\Theta = \theta) \frac{\beta - \rho_0}{\beta} \left( e^{-\rho_0 u} + \bar{F}(\theta) \right).
\]

The expression in (43) is illustrated in the following example.

**Example 15.** Let \( X \sim \text{Exp}(1) \) and \( \bar{F}_{W_1, W_2, \ldots, W_k}(x_1, \ldots, x_k) \) be defined as in (38) where \( C \) is an AMH copula. It means that \( \Theta \) follows a geometric distribution with parameter \( q = 1 - \alpha \) and \( L_{\Theta}(t) = \frac{qe^{-t}}{1 - (1 - q) e^{-t}} \) and \( L_{\Theta}^{-1}(u) = -\ln \left( \frac{q}{q + 1 - q} \right) \).

Then,

\[
\bar{F}_{W | \Theta = \theta}(x) = \left( \frac{1}{qe^x + 1 - q} \right)^\theta
\]

and

\[
f_{W | \Theta = \theta}(x) = \frac{\theta q e^x}{(qe^x + 1 - q)^{\rho_0 + 1}}
\]

for \( x \geq 0, \theta \in \mathbb{N} \) and \( 0 < q < 1 \). Note that \( \bar{F}_{W | \Theta = \theta}(x) = \frac{1}{q e^x + 1 - q} \) corresponds to the univariate survival function of the exponential distribution with tilt as defined in Marshall and Olkin (2007).

In this case, we have

\[
\mathbb{E}[W | \Theta = \theta] = \int_0^\infty \left( \frac{1}{q e^x + 1 - q} \right)^\theta dx = \lim_{t \to \infty} \int_0^t \left( \frac{1}{q e^x + 1 - q} \right)^\theta dx.
\]

Let \( u = qe^x + 1 - q \), then (44) becomes

\[
\int_0^\infty \left( \frac{1}{q e^x + 1 - q} \right)^\theta dx = \lim_{t \to \infty} \int_0^t \frac{1}{u^\theta (u - 1 + q) du}.
\]

Using a partial fraction decomposition, we obtain

\[
\int_0^\infty \left( \frac{1}{q e^x + 1 - q} \right)^\theta dx = \frac{\ln(q)}{(1 - q)^\theta} - \sum_{k=1}^{(\theta - 1)} \frac{1}{(\theta - k)(1 - q)^k}.
\]
6. Classical discrete-time risk models with exchangeable losses

In this section, we consider a discrete-time risk model with exchangeable losses. Let \( X = \{X_k, k \in \mathbb{N}\} \) be a sequence of exchangeable rvs, where \( X_k \) is the aggregate loss for a portfolio in period \( k \in \mathbb{N} \) with \( X_k \sim X_k, k \in \mathbb{N} \). Let \( U = \{U_k, k \in \mathbb{N}_0\} \) be the surplus process of the portfolio, where \( U_k \) corresponds to the surplus level at period \( k \in \mathbb{N} \). For \( k = 0 \), \( U_0 = u \) corresponds to the initial amount of capital allocated to the portfolio. Then, at period \( k \in \mathbb{N} \), \( U_k = U_{k-1} + \pi - X_k = u - \sum_{j=1}^{k} (X_j - \pi) \). The time of ruin \( \tau_u = \inf\{k \in \mathbb{N}, U_k < 0\} \), if \( U \) goes below 0 at least once, or \( \infty \), if \( U \) never goes below 0. We define the infinite-time ruin probability by \( \zeta(u) = \Pr(\tau_u < \infty) \). To prevent ruin with certainty \((i.e., \zeta(u) = 1, u \geq 0)\), we assume that the net profit condition is satisfied, i.e., \( E[X - \pi] < 0 \), where \( \pi \) is the premium income per period with \( \pi = (1 + \eta)E[X] \). For simplification purposes, we assume \( X \in \mathbb{N}_0, u \in \mathbb{N}_0 \), and \( \pi = 1 \) with \( \pi > E[X] \). Then, with these additional assumptions, the classical discrete-time risk model with exchangeable losses corresponds to an extension of the compound binomial classical risk model (see e.g. Gerber, 1988; Shiu, 1989; Willmot, 1993; De Vylder and Marcoux, 1996; Dickson, 1992 for details on the compound binomial risk model).

For \( j = 2, 3, \ldots \), the multivariate distribution of \((X_1, \ldots, X_j)\) is defined with an Archimedean copula \( C \) with either (18) or (19) as in Section 2.3. Let \( \zeta_0(u) \) be the conditional infinite-time ruin probability given \( \theta = \theta \). There exists a \( \theta_0 \) such that \( E[X|\theta = \theta] < 1 \), for \( \theta \in \{1, 2, \ldots, \theta_0\} \), and \( E[X|\theta = \theta] > 1 \), for \( \theta \in \{\theta_0 + 1, \theta_0 + 2, \ldots\} \). Then, when \( \theta = \theta_0 + 1, \theta_0 + 2, \ldots \), the solvency condition is not satisfied and the conditional infinite-time ruin probability \( \zeta_0(u) \) is 1, for all \( u \in \mathbb{N}_0 \). For \( \theta = 1, 2, \ldots, \theta_0 \), adapting expressions from Cossette et al. (2003), we have

\[
\zeta_0(u) = \frac{\zeta_0(u-1) - \sum_{j=1}^{u} \zeta_0(u-j) \times f_{X|\theta=\theta}(j) - F_{X|\theta=\theta}(u)}{f_{X|\theta=\theta}(0)},
\]

for \( u \in \mathbb{N}_0 \),

with initial value \( \zeta_0(0) = \frac{E[X|\theta=\theta] - \Pr(X<0|\theta=\theta)}{f_{X|\theta=\theta}(0)} \). The unconditional infinite-time ruin probability \( \zeta(u) \) is given by

\[
\zeta(u) = \sum_{\theta=1}^{\infty} \zeta_0(u)f_{\theta}(\theta) = \sum_{\theta=1}^{\theta_0} \zeta_0(u)f_{\theta}(\theta) + F_{\theta}(\theta_0), \quad u \in \mathbb{N}_0.
\]

Example 16. Let \( X \) be a non-negative discrete rv with

\[
f_X(0) = (1 - \delta) + \delta \times f_X(0)
\]

and \( f_X(k) = \delta \times f_X(k), k \in \mathbb{N}, \delta \in (0, 1) \), and \( B = 1 - NB(r, q) \) \((r \in \mathbb{R}^+, q \in (0, 1))\) with \( E[B] = 1 + r \times \frac{q}{1-q} \). We fix the different parameters as follows: \( \delta = 0.1, r = 2, q = 0.5 \), and \( E[X] = 0.9 < 1 \). Finally, \( F_{X|\theta=\theta} \) is defined with an AMH copula as in (18). The ruin probability is calculated for several values of initial capital and different values of dependence parameter \( \alpha \). Results are presented in Fig. 4 from which we can see that for a small dependency parameter, the ruin probability tends to zero. Otherwise, the greater the parameter becomes, the more likely the probability of ruin tends to a value close to 0.35. Once again, we emphasize the significant impact of a low to moderate dependence relation between rvs on the overall portfolio.

In the following example we provide an analytical expression of \( \zeta \) for a specific loss distribution.

Example 17. Let \( X \) be a non-negative discrete rv with

\[
f_X(0) = (1 - \delta) \quad \text{and} \quad f_X(k) = \delta, \quad k \in \mathbb{N}, \delta \in (0, 0.5)\).
\]

Assume that \( F_{X|\theta=\theta} \) is defined by (18) with copula \( C \). Using (22), we find

\[
f_{X|\theta=\theta}(0) = 1 - \delta_0 = e^{-\theta_0 C^{-1}(0)}.
\]

From Example 3.1 of Willmot (1993), we find

\[
\zeta_0(u) = \left(\frac{\delta_0}{1 - \delta_0}\right)^{u+1}, \quad (47)
\]

for \( u \in \mathbb{N}_0 \) and \( \theta = 1, 2, \ldots, \theta_0 \). Replacing (47) in (46), the expression \( \zeta(u) \) becomes

\[
\zeta(u) = \sum_{\theta=1}^{\theta_0} f_{\theta}(\theta) \left(\frac{\delta_0}{1 - \delta_0}\right)^{u+1} + F_{\theta}(\theta_0), \quad u \in \mathbb{N}_0.
\]

For illustration purposes, we fix the different parameters as follows: \( \delta = 0.4 \), implying \( E[X] = 0.8 < 1 \). Finally, \( F_{X|\theta=\theta} \) is defined by (18) where \( C \) is a Frank copula with \( \alpha = 4 \). We obtain that \( \theta_0 = 9 \) and

\[
\zeta(u) = \sum_{\theta=1}^{9} \left(1 - e^{-4}\right)^{\theta} \left(\frac{\delta_0}{1 - \delta_0}\right)^{u+1} + F_{\theta}(9), \quad u \in \mathbb{N}_0,
\]

where the values of \( \delta_0 \) are given in Table 22.

7. Partially nested Archimedean copulas

In the present section, we generalize the proposed strategy of Section 2 and adapt it to hierarchical structures for which at least one of the arguments is an Archimedean copula. More precisely, we consider in detail nested Archimedean copulas. Since there are several ways of nesting copulas and the computational methodology based on the mixing representation is the same for any chosen structure, we will only present a detailed example where we apply the strategy to a partially nested Archimedean copula. See Joe (1997), McNeil (2008) and Hofert (2011) for a general introduction to nested Archimedean copulas.

Let \( C \) be a one level partially nested Archimedean copula with \( d \) children, i.e., \( C \) is of the form

\[
C(u) = C(u; \psi_0, \psi_1, \ldots, \psi_d) = C(C(u_1; \psi_1), \ldots, C(u_d; \psi_d); \psi_0)
\]

\[
= C(C(u_1, \ldots, u_1.n_1; \psi_1), \ldots, u_1.n_1, \psi_d; \psi_0), \quad (48)
\]

\[\begin{align*}
\text{with} \quad (\psi_0, \psi_1, \ldots, \psi_d) = (\varphi_0, \varphi_1, \ldots, \varphi_d) = (\varphi_0, \varphi_1, \ldots, \varphi_d).
\end{align*}\]
where \( \psi \) is a strictly positive discrete rv defined on \( \mathbb{N} \) with pmf \( f_{\psi,j}(\theta_k) = \Pr(\theta_k = j) \) and cdf \( F_{\psi,j}(\theta_k) = \Pr(\theta_k \leq j) = \sum_{j=0}^{\infty} f_{\psi,j}(j) \) for \( \theta_k \in \mathbb{N} \) and \( i = 1, 2, \ldots, d \). Then, (50) becomes

\[
F_X(k_1 h, \ldots, k_d h) = \prod_{\theta_k=1}^{d} \left( \frac{\sum_{j=0}^{n_i} \sum_{j=1}^{\infty} f_X(\theta_k, \psi_{\theta_k,1} = j, (k_i h) f_{\psi,j}(\theta_k, i))}{\prod_{\theta_k=1}^{d} \left( \sum_{j=0}^{n_i} \sum_{j=1}^{\infty} f_X(\theta_k, \psi_{\theta_k,1} = j, (k_i h) f_{\psi,j}(\theta_k, i)) \right) \right) ,
\]

(51)

We define \( S = \sum_{i=1}^{d} \sum_{j=1}^{n_i} X_{i,j} = \sum_{i=1}^{d} S_i \) where \( S_i = \sum_{j=1}^{n_i} X_{i,j} \)

for \( i = 1, \ldots, d \). Then, similarly to (27), we have

\[
f_{S_i|\theta_{0,i} = \theta_0}(k) = \sum_{\theta_i=1}^{d} f_{S_i|\theta_{0,i} = \theta_0, \psi_{\theta_{0,i}} = k}(k) f_{\psi,j}(\theta_i, i) ,
\]

for \( i = 1, \ldots, d \),

and

\[
f_S(k) = \sum_{\theta_0=1}^{d} f_S(k) f_{\theta_0}(\theta_0) ,
\]

where

\[
(S|\theta_0 = \theta_0) = \sum_{i=1}^{d} (S_i|\theta_0 = \theta_0) ,
\]

and

\[
(S|\theta_0 = \theta_0, \theta_{0,i} = \theta_{0,i}) = \sum_{i=1}^{d} (X_{i,j}|\theta_0 = \theta_0, \theta_{0,i} = \theta_{0,i}) .
\]

Note that \((S_1|\theta_0 = \theta_0) = \ldots, (S_d|\theta_0 = \theta_0)\) are conditionally independent. Within each class \( i = 1, \ldots, d \), \((X_{i,j}|\theta_0 = \theta_0, \theta_{0,i} = \theta_{0,i})\) are also conditionally independent. It implies that

\[
f_{S_i|\theta_0 = \theta_0, \theta_{0,i} = \theta_{0,i}}(k) = f_{S_i|\theta_0 = \theta_0, \psi_{\theta_0,1} = k}(k) \cdot f_{\psi,j}(\theta_i, i) ,
\]

(52)

and

\[
f_S(k) = f_{S_1|\theta_0 = \theta_0, \psi_{\theta_0,1} = k}(k) \cdot \ldots \cdot f_{S_d|\theta_0 = \theta_0, \psi_{\theta_0,1} = k}(k) ,
\]

(53)

for \( k \in \mathbb{N} \) and \( i = 1, \ldots, d \) and where “\( \psi \)” denotes the convolution product. Values of (52) and (53) are computed using the same tools (e.g. DePril, FFT, etc.) as the ones mentioned in Section 2.

**Algorithm 18.** Let \( \theta_0 \in [1, 2, \ldots, \theta^*] \).

1. Begin with \( \theta_0 = 1 \).
2. For each child copula \( C_i \) where \( i \in \{1, 2, \ldots, d\} \), let \( \theta_{0,i} \in \{1, 2, \ldots, \theta^*_i\} \) and proceed as follows:
   (a) Begin with \( \theta_{0,i} = 1 \).
   (b) For \( j = 1, \ldots, n_i \), calculate \( F_{X_{i,j}|\theta_{0,i} = \theta_{0,i}, \psi_{\theta_{0,i},1} = k_{i,j}}(k_{i,j}) = e^{-\theta_{0,i} \psi_{\theta_{0,i},1}^{-1}(f_{X_{i,j}|\theta_{0,i} = \theta_{0,i}, \psi_{\theta_{0,i},1} = k_{i,j}}(k_{i,j}))} \) for \( k_{i,j} \in \mathbb{N} \).
   (c) For \( j = 1, \ldots, n_i \), calculate

\[
F_{X_{i,j}|\theta_{0,i} = \theta_{0,i}, \psi_{\theta_{0,i},1} = k_{i,j}}(k_{i,j})
\]
Example 19. Consider a portfolio of 80 risks \( X = (X_1, \ldots, X_{1,40}, X_{2,1}, \ldots, X_{2,40}) \) with multivariate cdf defined as in (51) with \( d = 2 \) and \( n_1 = n_2 = 40 \). Assume \( C_{0,0}, C_{0,1}, \ldots, C_{d,d} \) to be AMH copulas, with respective parameters \( \alpha_0, \alpha_1, \ldots, \alpha_d \) (with \( 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_d \)), then it implies that \( \Theta \sim \text{Geo}(1/\alpha_0) \) and \( \Theta \sim \text{Shifted NB}(1/\alpha_0, 1/\alpha_0) \), for \( i = 1, \ldots, d \), as shown in Hofert (2010) (see details in Appendix). In the following example, we consider a two-dimensional partially nested Archimedean copula defined as in (48), where all copulas involved are AMH copulas. The example illustrates the accuracy of the proposed strategy in comparison to the MC simulation method.

Example 20. Assume a multivariate cdf of \( X = (X_1, X_2, X_3, X_4, X_5) \) defined with binomial marginals and the following five-dimensional partially nested Archimedean copula:

\[
C(u_1, u_2, u_3, u_4, u_5) = C_{0,0} \left( C_{0,1} (u_1, u_2), C_{0,2} (u_3, C_{0,3} (u_4, u_5)) \right),
\]

where \( C_{0,j} \), \( j = 0, 1, 2, 3 \), correspond to bivariate AMH copulas. Also, \( X_i \sim \text{Bin} (10, 0.05) \), i.e. \( \alpha_1 = 1.2, \ldots, 5, \) and \( \alpha_0 = 0.1j + 0.2, j = 0, 1, 2, 3 \). Let \( S = \sum_{i=1}^{5} X_i \) which implies \( E[S] = 7.5 \). In Table 24, we provide the exact and the simulated (10 million MC simulations) values of \( f_{2}(k) \), for \( k = 0, 1, 2, \ldots, 50 \) in addition to their confidence intervals (with a confidence level of 95%). We noticed a fluctuation in the results for several distinct 10 million simulation paths, contrarily to our proposed approach.

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Appendix A. Archimedean copulas

Archimedean copulas defined with a strictly positive discrete mixing rv \( \Theta \):

1. Ali–Mikhail–Haq (AMH) family:
   - Copula:
     \[ C_{\Theta} (u_1, \ldots, u_d) = (1 - \alpha) (\prod_{i=1}^{d} (1 - u_i) + \alpha)^{-1} \]
   - Parameter: \( \alpha \in (0, 1) \)
   - Discrete distribution for \( \Theta \): Shifted Geometric(1 - \alpha)
   - Pmf: \( f_\Theta(k) = \alpha^{k-1} (1 - \alpha), \; k \in \mathbb{N} \)
   - LST: \( L_\Theta(t) = \frac{1 - \alpha}{\alpha - \alpha t} \)
   - Inverse of LST: \( L_\Theta^{-1}(u) = \frac{1 - \alpha}{\alpha - \alpha u} \)
   - Particular cases: as \( \alpha \to 0 \), \( C_{\Theta} (u_1, \ldots, u_n) = C^+ (u_1, \ldots, u_n) \)

2. Frank family:
   - Copula:
     \[ C_{\Theta} (u_1, \ldots, u_n) = \frac{1}{\alpha} \ln \left( 1 - \frac{1-e^{-\alpha u}}{1-e^{-\alpha u_1}} \cdot \ldots \cdot \frac{1-e^{-\alpha u_n}}{1-e^{-\alpha u_{n-1}}} \right) \]
   - Parameter: \( \alpha \in (0, \infty) \)
   - Discrete distribution for \( \Theta \): Logarithmic(1 - \alpha)
   - Pmf: \( f_\Theta(k) = \frac{1}{1-e^{-\alpha k}}, \; k \in \mathbb{N} \)
   - LST: \( L_\Theta(t) = -\frac{1}{\alpha} \ln \left( 1 - (1 - e^{-\alpha})^{-1} \right) \)
   - Inverse of LST: \( L_\Theta^{-1}(u) = -\ln \left( 1 - (1 - e^{-\alpha})^{-1} \right) \)
   - Particular cases: \( C_{\Theta=0} (u_1, \ldots, u_n) = C^+ (u_1, \ldots, u_n) \)
   - \( C_{\Theta=\infty} (u_1, \ldots, u_n) = C^{-1} (u_1, \ldots, u_n) \)

3. Shifted Negative Binomial family:
   - Copula: \( C_{\Theta} (u_1, \ldots, u_n) = \left( q \left( \prod_{i=1}^{d} (1 - u_i) + 1 - q \right) - (1 - q) \right)^{-1} \)
   - Parameter: \( r \in \mathbb{R}^+ \) and \( q \in (0, 1) \)
   - Discrete distribution for \( \Theta \): Shifted Negative Binomial(\( r, q \))
   - Pmf: \( f_\Theta(k) = \binom{r-q}{k} q^k (1 - q)^{r-k}, \; k = r, r + 1, \ldots \)
   - LST: \( L_\Theta(t) = \left( \frac{1-q e^{-t}}{1-(1-q) e^{-t}} \right)^{-r} \)
Appendix B. Nested Archimedean copula

Archimedean copulas defined with a strictly positive continuous mixing rv \( \Theta \):

1. Clayton family:
   - Copula:
     \[ C_{\alpha}(u_1, \ldots, u_n) = \left( u_1^{-\alpha} + \cdots + u_n^{-\alpha} - (n-1)^{-\alpha} \right)^{-\frac{1}{\alpha}} \]
   - Parameter: \( \alpha \in (0, \infty) \)
   - Continuous distribution for \( \Theta \): Gamma(\( \frac{1}{\alpha}, 1 \))
   - LST: \( L_\alpha(t) = \left( \frac{t}{1+t} \right)^{\frac{1}{\alpha}} \)
   - Inverse of LST: \( L^{-1}_\alpha(u) = \left( \frac{u-1}{u} \right)^{-\alpha} + 1 \)
   - Particular cases: \( C_{\alpha=0}(u_1, \ldots, u_n) = C^+(u_1, \ldots, u_n) \) and \( C_{\alpha-\infty}(u_1, \ldots, u_n) = C^-(u_1, \ldots, u_n) \)

2. Gumbel family:
   - Copula: \[ C_{\alpha}(u_1, \ldots, u_n) = \exp \left( -(-\ln(u_1)^{\alpha} + \cdots + (-\ln(u_n)^{\alpha})^{\frac{1}{\alpha}} \right) \]
   - Parameter: \( \alpha \in (1, \infty) \)
   - Continuous distribution for \( \Theta \): Positive Stable (\( \frac{1}{\alpha}, 1, \cos(\alpha(\frac{1}{2})) \), 1(\( \alpha=1 \))
   - LST: \( L_\alpha(t) = e^{-e^{-\alpha t}} \)
   - Inverse of LST: \( L^{-1}_\alpha(u) = (-\ln(u))^\alpha \)
   - Particular cases: \( C_{\alpha-1}(u_1, \ldots, u_n) = C^+(u_1, \ldots, u_n) \) and \( C_{\alpha-\infty}(u_1, \ldots, u_n) = C^-(u_1, \ldots, u_n) \).

\[ \text{Table 24} \]

<table>
<thead>
<tr>
<th>( k )</th>
<th>( f_s(k) ) (exact values)</th>
<th>( f_s(k) ) (simulated values)</th>
<th>( K_{\Theta,05} )</th>
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<tr>
<td>0</td>
<td>0.000808</td>
<td>0.000805</td>
<td>[0.0000786; 0.000821]</td>
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<tr>
<td>1</td>
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<td>0.005785</td>
<td>[0.0055714; 0.005808]</td>
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<td>2</td>
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<td>0.020082</td>
<td>[0.019991; 0.020165]</td>
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<td>3</td>
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<td>0.045699</td>
<td>[0.0455589; 0.045856]</td>
</tr>
<tr>
<td>4</td>
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<td>0.078477</td>
<td>[0.078137; 0.078470]</td>
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<td>5</td>
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<td>0.108730</td>
<td>[0.108657; 0.109043]</td>
</tr>
<tr>
<td>10</td>
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<td>0.086248</td>
<td>[0.086074; 0.086421]</td>
</tr>
<tr>
<td>15</td>
<td>0.060728</td>
<td>0.060732</td>
<td>[0.060666; 0.060768]</td>
</tr>
</tbody>
</table>

\[ \text{Table 25} \]

<table>
<thead>
<tr>
<th>( \bar{\theta} )</th>
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<tr>
<td>( \epsilon )</td>
<td>( \epsilon )</td>
<td>( \epsilon )</td>
<td>( \epsilon )</td>
</tr>
</tbody>
</table>

Distribution of \( \Theta_0 \): \( \Theta_0 \sim \Theta_{\Theta,05} \) with \( \Theta_0 \leq \Theta_{\Theta,05} \)

Distribution of \( \Theta_0 \): \( \Theta_0 \sim \Theta_{\Theta,05} \) with \( \Theta_0 \leq \Theta_{\Theta,05} \)

References
