On two families of bivariate distributions with exponential marginals: Aggregation and capital allocation

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ABSTRACT

In this paper, we consider two main families of bivariate distributions with exponential marginals for a couple of random variables \((X_1, X_2)\). More specifically, we derive closed-form expressions for the distribution of the sum \(S = X_1 + X_2\), the TVaR of \(S\) and the contributions of each risk under the TVaR-based allocation rule. The first family considered is a subset of the class of bivariate combinations of exponentials, more precisely, bivariate combinations of exponentials with exponential marginals. We show that several well-known bivariate exponential distributions are special cases of this family. The second family we investigate is a subset of the class of bivariate mixed Erlang distributions, namely bivariate mixed Erlang distributions with exponential marginals. For this second class of distributions, we propose a method based on the compound geometric representation of the exponential distribution to construct bivariate mixed Erlang distributions with exponential marginals. Notably, we show that this method not only leads to Moran–Downton’s bivariate exponential distribution, but also to a generalization of this bivariate distribution. Moreover, we also propose a method to construct bivariate mixed Erlang distributions with exponential marginals from any absolutely continuous bivariate distributions with exponential marginals. Inspired from Lee and Lin (2012), we show that the resulting bivariate distribution approximates the initial bivariate distribution and we highlight the advantages of such an approximation.

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1. Introduction

Results related to the sum of dependent risks are of great interest not only in actuarial science, quantitative risk management, statistics and applied probability in general. The calculation of the overall capital charge for a portfolio of risks, the evaluation and analysis of risk measures for decision making, and strategic planning require the knowledge of the cumulative distribution function (cdf) of the sum of dependent random variables (rvs). Risk measures, such as the Value-at-Risk (VaR) and the Tail-Value-at-Risk (TVaR), can be used to evaluate this total capital requirement. Artzner et al. (1999) proposed the TVaR, also called Expected Shortfall (ES) in quantitative risk management, as a coherent alternative to the non-coherent risk measure VaR. It is also of interest to determine the amount of allocated capital for each risk within the portfolio. To perform that task, a top-down approach can be used in which an amount of capital is computed for the entire portfolio and a capital allocation method is then chosen to quantify the contribution of each risk (see e.g. Tasche, 1999 and McNeil et al., 2005 for details on capital allocation rules).

Over the last decade, the study of the aggregation of dependent risks and the top down allocation method introduced by Tasche (1999) has led to several closed-form formulas for the distribution of aggregated risks, its TVaR and the TVaR-based allocations. Such an allocation method allows to take into consideration the dependence structure between the components of the portfolio. Researchers have investigated this problem under different types of multivariate continuous distributions: multivariate normal distribution (Panjer, 2002), multivariate elliptical distribution (Landsman and Valdez, 2003; Dhaene et al., 2008), multivariate gamma distribution (Furman and Landsman, 2005), multivariate Tweedie distribution (Furman and Landsman, 2008), and multivariate Pareto distribution (Chiragiev and Landsman, 2007). Bargès et al. (2009) gave a closed-form expression for the TVaR-based allocation when lines of business of an insurance portfolio are linked with a Farlie–Gumbel–Morgenstern (FGM) copula and when marginal risks are distributed as mixtures of exponentials. Extending these results, Cossette et al. (2013) investigated risk aggregation and capital allocation problems for a
portfolio of dependent risks whose multivariate distribution is defined with the Farlie–Gumbel–Morgenstern copula and mixed Erlang marginals. Cossette et al. (2012) examined the computation of the TVaR and the TVaR-based allocation for multivariate compound distributions and obtained closed-form expressions.

Bivariate exponential distributions play an important role in actuarial science, statistics, reliability, survival modeling, applied probability, and quantitative risk management. A recent survey on bivariate exponential distributions can be found in Balakrishnan and Lai (2009). See also e.g. Kotz et al. (2004). In the present paper, two families of absolutely continuous bivariate distributions with exponential marginals are considered for a couple of random variables \((X_1, X_2)\). For these two families of bivariate distributions, we develop closed-form expressions for the distribution of the sum \(S = X_1 + X_2\), the TVaR of \(S\) and the contributions of each risk under the TVaR-based allocation rule. The first family is a subset of the class of bivariate combinations of exponentials, more precisely, bivariate combinations of exponentials with exponential marginals. We show that several well-known bivariate exponential distributions are special cases of this family, notably the bivariate FGFM exponential distribution and the bivariate Bladt–Nielsen exponential distribution. The second family is a subset of the class of bivariate mixed Erlang distributions, namely bivariate mixed Erlang distributions with exponential marginals. For this second class of distributions, we propose a method based on the compound geometric representation of the exponential distribution to construct bivariate mixed Erlang distributions with exponential marginals. Notably, we show that this method not only leads to Moran–Downton’s bivariate exponential distribution, but also to a generalization of this bivariate distribution. Moreover, we also propose a method to construct bivariate mixed Erlang distributions with exponential marginals from any absolutely continuous bivariate distributions with exponential marginals. Inspired from Lee and Lin (2012), we show that the resulting bivariate distribution approximates the initial bivariate distribution and we discuss the advantages of such an approximation.

The outline of the paper is as follows. We briefly recall known elements in Section 2 which will be useful throughout the paper. We discuss bivariate exponential distributions in Section 3 which sets the table for Section 4 where the quantities of interest in regard to aggregation and capital allocation are studied within the family of bivariate combinations of exponential distributions. In Section 5, we pursue with the family of bivariate mixed Erlang distributions with exponential marginals.

2. Definitions and preliminary results

In this section, we present definitions and results needed throughout the paper. Let \((X_1, X_2)\) be a couple of continuous non-negative rvs with joint probability density function (pdf) \(f_{X_1X_2}\), joint cumulative distribution function \(F_{X_1X_2}\), and joint moment generating function (mgf) \(M_{X_1X_2}\). To identify the distribution of \(S = X_1 + X_2\) in the following sections, we use the pdf of \((X_1, X_2)\) with \(f_S(x) = \int_{\mathbb{R}^2} f_{X_1X_2}(y, x - y) \, dy\) or the mgf of \((X_1, X_2)\) with \(M_S(t) = M_{X_1X_2}(t, t)\). VaR and TVaR are probably the most widely used risk measures in insurance and financial institutions. The Value-at-Risk at level \(\kappa\), \(0 < \kappa < 1\), of a rv \(X\) is defined by \(\text{VaR}_\kappa(X) = \inf\{x \in \mathbb{R}, F_X(x) \geq \kappa\}\). The Tail-Value-at-Risk at level \(\kappa\), \(0 \leq \kappa < 1\), of the rv \(X\) is defined by

\[
\text{TVaR}_\kappa(X) = \frac{1}{1 - \kappa} \int_\kappa^1 \text{VaR}_\kappa(X) \, du
\]

which becomes

\[
\text{TVaR}_\kappa(X) = \frac{\mathbb{E}[X \times 1_{\{X > \text{VaR}_\kappa(X)\}}]}{1 - \kappa}.
\]

when \(X\) is a continuous rv (see e.g. McNeil et al., 2005 and Marceau, 2013 for details). Throughout the paper, only non-negative continuous rvs are considered. Also, note that \(\mathbb{E}[X \times 1_{\{X > \text{VaR}_\kappa(X)\}}] \) corresponds to the truncated expectation of the rv \(X\) where \(1_\kappa\) is the indicator function such that \(1_\kappa(X) = 1, \text{if } X \in A\), and \(1_\kappa(X) = 0\), i.e. \(\mathbb{E}[X \times 1_{\{X > \text{VaR}_\kappa(X)\}}] = \int_b^{\infty} x f_X(x) \, dx\).

In the following sections, we also derive analytical expressions for the contribution of the \(i\)th risk to the aggregate risk of the portfolio based on the TVaR allocation rule. It is crucial for an insurance company or a financial institution to evaluate the overall capital charge for a portfolio of risks in order to protect itself from large rare events. For a portfolio of non-negative continuous risks, according to Tasche (1999), the contribution of the risk \(X_i\) is defined as

\[
\text{TVaR}_\kappa(X_i; S) = \frac{\mathbb{E}[X_i \times 1_{\{S > \text{VaR}_\kappa(S)\}}]}{1 - \kappa}
\]

(see also e.g. McNeil et al., 2005 and Marceau, 2013 for details). For further use, we rewrite the numerator of the previous expression as follows

\[
\mathbb{E}[X_i \times 1_{\{S > \text{VaR}_\kappa(S)\}}] = \int_{\text{VaR}_\kappa(S)}^{\infty} g_{X_i,S}(s) \, ds,
\]

where \(g_{X_i,S}(s) = \int_s^{\infty} f_{X_i,X_j}(x, s - x) \, dx\). To simplify the presentation, we examine the contribution of \(X_1\), while the contribution of \(X_2\) is simple to find by similarity. It can be easily shown that the sum of the two TVaR contributions for each individual risk is equal to the capital required for the whole portfolio, i.e. \(\text{TVaR}(S) = \sum_{i=1}^{n} \text{TVaR}(X_i; S)\). Note that, given the continuous assumption for the rvs \(X_1\) and \(X_2\), (1) can also be written as

\[
\text{TVaR}_\kappa(X_i; S) = \mathbb{E}[X_i | S > \text{VaR}_\kappa(S)]
\]

3. Bivariate distributions with exponential marginals

Several bivariate distributions with exponential marginals were proposed in the literature, see e.g. Kotz et al. (2004) and Balakrishnan and Lai (2009) for a review. Here, we focus on two main families of bivariate distributions with exponential marginals. The first family is a subset of the class of bivariate combinations of exponentials, for which we investigate the distribution of the sum \(S = X_1 + X_2\), the Tail-Value-at-Risk \(\text{TVaR}(S)\) and the contributions under the TVaR-based allocation rule. We refer to this subset as bivariate combinations of exponentials with exponential marginals. The second family is a subset of the class of bivariate mixed Erlang distributions which we designate as bivariate mixed Erlang distributions with exponential marginals.

More precisely, in this paper, we assume that the couple of rvs \((X_1, X_2)\) has a bivariate distribution with exponential marginals with mean \(1/\mu_i\), for \(i = 1, 2\). We define the Fréchet class \(\Gamma_F\) of \(F_{X_1X_2}\) with exponential marginals \(F_{X_i}(x_i) = 1 - e^{-\mu_i x_i}\), \(i = 1, 2\). The elements of \(\Gamma_F\) are bounded above and below by the Fréchet upper and lower bounds, meaning \(F_{X_1X_2}(x_1, x_2) \leq F_{X_1X_2}(x_1, x_2) \leq F_{X_1X_2}(x_1, x_2), \) where \(F_{X_1X_2}(x_1, x_2) = \max \{F_{X_1}(x_1) + F_{X_2}(x_2) - 1, 0\}\) and \(F_{X_1X_2}(x_1, x_2) = \min \{F_{X_1}(x_1) \times F_{X_2}(x_2)\}\). Note that the Fréchet lower and upper bounds correspond respectively to countemerontonicty and comonotonicity. See e.g. Nelsen (2006) or Depridnuit et al. (2005) for details.

The Pearson correlation coefficient is a measure of association for two rvs that captures their level of linear correlation. It is defined as \(\rho_P (X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1) \times \text{Var}(X_2)}}\). Note that the bounds on \(\rho_P (X_1, X_2)\) are

\[
\rho_{\min} = 1 - \frac{\pi^2}{6} \leq \rho_P (X_1, X_2) \leq 1 = \rho_{\max}.
\]
where the upper (lower) bound $\rho_{\text{max}}$ ($\rho_{\text{min}}$) is attained when the components of $(X_1, X_2)$ are comonotonic (countermonotonic) (see e.g. Denuit et al., 2005, McNeil et al., 2005, and Bladt and Nielsen, 2010b). We will provide the expression of Pearson’s correlation coefficient for the different families of bivariate distributions considered to capture the degree of linear relationship within $(X_1, X_2)$.

One important member of the Fréchet class $\Gamma \left( F_{X_1}, F_{X_2} \right)$ corresponds to the case where $X_1$ and $X_2$ are independent. In the following proposition, we recall in this context expressions for the cdf of $S = X_1 + X_2$ and the expectation terms in the TVAR and risk allocation. These are well known results but we restate them to establish the notations. Also, these expressions will be useful throughout the paper.

**Proposition 1.** Let $X_1$ and $X_2$ be independent exponentially distributed rvs with mean $1/\beta_i$, $(i = 1, 2)$, with

$$f_{X_1, X_2}(x_1, x_2) = \beta_1 e^{-\beta_1 x_1} \beta_2 e^{-\beta_2 x_2},$$

and let $S = X_1 + X_2$. Then, we have

$$F_S(x) = H(x; \beta_1, \beta_2) = \begin{cases} 
1 - e^{-\beta_1} \sum_{j=0}^{2-1} \frac{(\beta x)^j}{j!}, & \beta_1 = \beta_2 = \beta \\
\sum_{j=1}^{2} \left( \frac{2}{j!} \beta \right)^j (1 - e^{-\beta x}), & \beta_1 \neq \beta_2,
\end{cases}$$

(3)

and

$$E[S \times 1_{[S-b]}] = \xi_3(b; \beta_1, \beta_2),$$

(4)

$$g_{S,1,S}(s) = \psi_1(s; \beta_1, \beta_2) \begin{cases} 
\frac{1}{\beta} h(x; 3, \beta), & \beta_1 = \beta_2 = \beta \\
\left( 1 - e^{-\beta x} \right) \left( \beta_1 - \beta_2 \right)^2, & \beta_1 \neq \beta_2,
\end{cases}$$

(5)

**Remark 2.** In (4), the rv $S$ follows an Erlang-2 distribution if $\beta_1 = \beta_2 = \beta$ and a generalized Erlang distribution if $\beta_1 \neq \beta_2$. Both expressions in (4) are provided in e.g. Gerber and Shiu (2005). The computation of the TVAR and the contribution based on the TVAR allocation rule for non-negative independent rvs are treated e.g. in Section 2 of Furman and Landsman (2005). They also discuss, in Sections 3 and 4, the particular case of the sum of independent gamma rvs.

**4. Bivariate combinations of exponentials with exponential marginals**

The class of bivariate combinations of exponential distributions with exponential marginals, denoted by $\Gamma^{CE} \left( F_{X_1}, F_{X_2} \right)$, is a subset of the larger class of bivariate combinations of exponentials. Obviously, the class $\Gamma^{CE} \left( F_{X_1}, F_{X_2} \right)$ is also a subset of the Frechet class $\Gamma \left( F_{X_1}, F_{X_2} \right)$ defined in Section 3 (i.e. $\Gamma^{CE} \left( F_{X_1}, F_{X_2} \right) \subseteq \Gamma \left( F_{X_1}, F_{X_2} \right)$). In this section, we first briefly present the general class of bivariate combinations of exponential distributions, including Proposition 4 which provides the general expressions for $F_S(x), E[S \times 1_{[S-b]}]$, and $E[X_1 \times 1_{[S-b]}]$. Then, we show that several well known bivariate exponential distributions are members of $\Gamma^{CE} \left( F_{X_1}, F_{X_2} \right)$, which allows us to apply Proposition 4 to derive the desired expressions for $F_S(x), E[S \times 1_{[S-b]}]$, and $E[X_1 \times 1_{[S-b]}]$ for those distributions.

**4.1. Bivariate combinations of exponentials**

Let $(X_1, X_2)$ follow a bivariate combination of exponential distributions with joint pdf given by

$$f_{X_1, X_2}(x_1, x_2) = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} e^{-\gamma_{1}^{i} x_1} e^{-\gamma_{2}^{j} x_2},$$

(6)

where $c_{ij} \in \mathbb{R}$ with $\sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} = 1$. We assume $0 < \gamma_{1}^{i} < \cdots < \gamma_{m}^{i}$ and $0 < \lambda_{1} < \cdots < \lambda_{m}$. Also, we denote $c_{i} = \sum_{j=1}^{m} c_{ij}$ and $c_{j} = \sum_{i=1}^{m} c_{ij}$ where $\{c_{ij}, i = 1, \ldots, m, j = 1, \ldots, m\}$ are such that $f_{X_1, X_2}(x_1, x_2) \geq 0$ for all $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$. Clearly, this class includes the class of bivariate mixed exponential distributions for which $0 \leq c_{ij} \leq 1$. Note that the class of bivariate combinations of exponential distributions are a subset within the family of bivariate matrix exponential distributions (see Bladt and Nielsen, 2010a about this family). As we shall see in what follows, there exist specific expressions for $c_{ij}$ for which the distribution to the distribution of $(X_1, X_2)$ are exponential.

Here, the marginals $F_{X_1}(x_1) = \sum_{i=1}^{m} c_{i} \left( 1 - e^{-\gamma_{1}^{i} x_1} \right)$ and $F_{X_2}(x_2) = \sum_{j=1}^{m} c_{j} \left( 1 - e^{-\gamma_{2}^{j} x_2} \right)$ are univariate combinations of exponentials. For example, for $m = 2$, we have $0 \leq c_{i} \leq \frac{\gamma_{1}^{i}}{\gamma_{1}^{2}},$ and $0 \leq c_{j} \leq \frac{\gamma_{2}^{j}}{\gamma_{2}^{2}}$. See e.g. Dufresne and Gerber (1988, 1991), Chan (1990) and Babier and Chan (1992) for applications of univariate combinations of exponentials in ruin theory and Langenbacher (2003) for applications in the context of pharmacetics and biopharmaceutics. Univariate combinations of exponential distributions are a subset of the family of matrix exponential distributions (see e.g. Asmussen and Bladt, 1997 and Dufresne, 2007 for details). Note that combinations of exponentials can also be called bi-exponentials or di-exponentials when $m = 2$ and polyexponentials when $m \geq 2$, see e.g. Babier and Chan (1992) or Langenbacher (2003).

**Proposition 4.** Let $(X_1, X_2)$ follow a bivariate combination of exponentials. Then, for $S = X_1 + X_2$, we have

$$F_S(x) = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} H(x; \gamma_{1}^{i}, \lambda_{1}),$$

(7)

and

$$E[S \times 1_{[S-b]}] = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} E[S \times 1_{[S-b]}],$$

(8)

**Proof.** The expressions are obtained straightforwardly from their definition. ■
and
\[
E \left[ X_1 \times 1_{[s,b]} \right] = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} \xi_1 \left( b; \gamma_1, \lambda_2 \right).
\]  
(9)

**Proof.** Given (6), the joint pdf of \((X_1, X_2)\) is a linear combination of \(m \times m\) terms where each one is of the form given in (3). Then, (7), (8), and (9) are obtained with Proposition 1. ■

**Remark 5.** From (7), it is clear that \(S\) follows a combination of Erlang-2 and/or generalized Erlang distributions.

**Remark 6.** The value of \(Var_r(S)\) is obtained by inverting \(F_3(x)\) using numerical methods. Then, we compute \(TVar_r(S) = \frac{E[S \times 1_{(S, Var_r(S))}] - Var_r(S)}{1 - k}\) and \(TVar_r(X_i; S) = \frac{E[X_i \times 1_{(S, Var_r(S))}] - Var_r(S)}{1 - k}\), for \(l = 1, 2\).

4.2. Examples of bivariate combinations of exponentials with exponential marginals

We consider some well known bivariate exponential distributions which belong to \(G^{CE}(F_{x_1}, F_{x_2})\).

**4.2.1. Bivariate distribution defined with FGM copula and exponential marginals**

Let the joint cdf of \((X_1, X_2)\) be defined with a Farlie–Gumbel–Morgenstern (FGM) copula
\[
C_\theta (u_1, u_2) = u_1 u_2 + \theta u_1 u_2 \left( 1 - u_1 \right) \left( 1 - u_2 \right), \quad -1 \leq \theta \leq 1,
\]  
(10)

(see e.g. Nelsen, 2006) and with exponential marginals with means \(1/\beta_1\) and \(1/\beta_2\). This leads to
\[
F_{X_1, X_2} (X_1, x_2) = C \left( F_{X_1} (x_1), F_{X_2} (x_2) \right) = \left( 1 - e^{-\beta_1 x_1} \right) \left( 1 - e^{-\beta_2 x_2} \right) + \theta \left( 1 - e^{-\beta_1 x_1} \right) \left( 1 - e^{-\beta_2 x_2} \right) e^{-\beta_1 x_1} e^{-\beta_2 x_2}. \quad (11)
\]

As mentioned in e.g. Johnson et al. (1997), it is also called Gumbel’s bivariate distribution. The expression for the joint pdf of \((X_1, X_2)\) is given by
\[
f_{X_1, X_2} (x_1, x_2) = \beta_1 e^{-\beta_1 x_1} \beta_2 e^{-\beta_2 x_2} + \theta \sum_{i=1}^{2} \sum_{j=1}^{2} (1-\theta)^{i+j}
\times \left( \frac{l}{m} \right) \left( \frac{k}{m} \right) \prod_{i=1}^{m} \prod_{j=1}^{m} \prod_{l=1}^{m} \prod_{k=1}^{m} p_{i,j} \left( -1 \right)^{-i-j} \times \left( \frac{l-1}{m-i} \right) \left( \frac{k-1}{m-j} \right). \quad (12)
\]

**4.2.2. Bladt–Nielsen’s bivariate exponential distribution**

Bladt and Nielsen (2010b) use multivariate phase-type distributions to define a class of bivariate exponential distributions with any feasible Pearson correlation coefficient \(\rho \left( X_1, X_2 \right) \in \left[ \rho_{\text{min}}, \rho_{\text{max}} \right]\). The characteristics of Bladt–Nielsen’s bivariate exponential distribution are summarized in the following theorem.

**Theorem 7.** Assume that \(m \in \mathbb{N}^+\) is fixed and that \(\rho \left( X_1, X_2 \right) = \rho \in \left[ \rho_{\text{min}}, \rho_{\text{max}} \right]\), where \(\rho_{\text{min}} = 1 - \sum_{k=1}^{m} \frac{1}{k^2}\) and \(\rho_{\text{max}} = 1 - \frac{1}{m!} \sum_{k=1}^{m} \frac{1}{k!}\). Then, the expression of the joint pdf of \((X_1, X_2)\) for Bladt–Nielsen’s bivariate exponential distribution is given by
\[
f_{x_1, x_2} (x_1, x_2) = \sum_{l=1}^{m} \sum_{k=1}^{m} c_{l,k} \lambda_l e^{-\lambda_l x_1} \lambda_k e^{-\lambda_k x_2}, \quad (13)
\]

where
\[
c_{l,k} = \left( \frac{-1}{{m \choose l}} \right) \left( \frac{m}{l} \right) \left( \frac{k}{m} \right) \prod_{i=1}^{m} \prod_{j=1}^{m} \prod_{l=1}^{m} \prod_{k=1}^{m} p_{i,j} \left( -1 \right)^{-i-j} \times \left( \frac{l-1}{m-i} \right) \left( \frac{k-1}{m-j} \right). \quad (14)
\]

and
\[
p_{i,j} = \left( \frac{\rho_{\text{max}}}{\rho_{\text{max}}} \right)^{i+j-n-1} + \frac{1}{m} \left( 1 - \frac{\rho_{\text{max}}}{\rho_{\text{max}}} \right), \quad \rho > 0
\]

\[
p_{i,j} = \left( \frac{\rho_{\text{min}}}{\rho_{\text{max}}} \right)^{i+j-n-1} + \frac{1}{m} \left( 1 - \frac{\rho_{\text{min}}}{\rho_{\text{max}}} \right), \quad \rho < 0,
\]

with \(\delta_x = 1\), if \(x = 0\).

**Proof.** See Theorems 4.1 and 4.4 of Bladt and Nielsen (2010b). ■

**Remark 8.** By Theorems 4.2 and 4.4 of Bladt and Nielsen (2010b), \(\lim_{m \to \infty} \rho_{\text{min}} = \rho_{\text{min}}\) and \(\lim_{m \to \infty} \rho_{\text{max}} = \rho_{\text{max}}\), respectively.

Note that for a specific \(m\), Bladt–Nielsen’s bivariate exponential distribution has 3 parameters: the rate parameters \(\beta_1\) and \(\beta_2\), and the dependence parameter \(\rho = \rho \left( X_1, X_2 \right)\). It can also be seen as a bivariate combination of exponentials with exponential marginals, \(F_{X_1, X_2} \in G^{CE}(F_{x_1}, F_{x_2})\). Indeed, given (13), we have \(\gamma_i = i \beta_1 (i = 1, 2, \ldots, m)\) and \(\lambda_j = j \beta_2 (j = 1, 2, \ldots, m)\). The coefficients \(c_{l,k}\) are provided in (14). With Proposition 4, we find the desired quantities.
Compared to the other examples of bivariate exponential distributions considered in this sub-section, Bladt–Nielsen’s bivariate exponential distribution can take into account any values of Pearson’s coefficient between $\rho_{\min}$ and $\rho_{\max}$. It is worth to mention that this distribution has no singularities.

4.2.3. Bivariate distribution defined with the AMH copula and exponential marginals

Let the joint cdf of $(X_1, X_2)$ be defined by a bivariate Ali–Mikhail–Haq (AMH) copula, given by

$$
C_\theta (u_1, u_2) = \frac{u_1 u_2}{1 - \theta (1 - u_1) (1 - u_2)} = u_1 u_2 + u_1 u_2 \sum_{k=1}^{\infty} \theta^k (1 - u_1)^k (1 - u_2)^k,
$$

with dependence parameter $\theta \in [-1, 1]$. As a special case, $C_0 (u_1, u_2) = u_1 u_2$ is the independence copula. The AMH copula is also a member of the Archimedean family of copulas. It introduces a moderate, positive or negative dependence relation. It is considered as a perturbation of the independence copula. The first-degree approximation of the AMH copula corresponds to the FGM copula (see e.g. Nelsen, 2006). With $F_{X_1, X_2} (x_1, x_2) = C_\theta (1 - e^{-\beta_1 x_1}, 1 - e^{-\beta_2 x_2})$ and using (15), we derive the expression for the joint pdf of $(X_1, X_2)$ which is given by

$$
f_{X_1, X_2} (x_1, x_2) = \beta_1 e^{-\beta_1 x_1} \beta_2 e^{-\beta_2 x_2} + \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} (k + i) \beta_1 e^{-(k+i)\beta_1 x_1} \times (k + j) \beta_2 e^{-(k+j)\beta_2 x_2}.
$$

The Pearson correlation coefficient is $\rho_F (X_1, X_2) = \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} (k + i) \beta_1 e^{-(k+i)\beta_1 x_1} \times (k + j) \beta_2 e^{-(k+j)\beta_2 x_2},$ with bounds $\min (16) \leq \rho_F (X_1, X_2) \leq 3$.

Clearly, given (16), $F_{X_1, X_2} \in \mathcal{C}^\epsilon (F_{X_1}, F_{X_2})$. Indeed, given (16), we have $m \rightarrow \infty, \gamma_1 = ip_1 (i \in \mathbb{N}^+), \gamma_2 = jp_2 (j \in \mathbb{N}^+), c_{i1} = (1 + \theta), c_{i2} = c_{i1} = -\theta, c_{i1} = 0, j = 2, 3, \ldots, c_{i1} = 1$, $i = 2, 3, \ldots, k = 2, 3, \ldots, c_{k1} = c_{k+1, k+1} = \theta, c_{k, k+1} = -\theta, c_{k1} = c_{k+1, j} = 0, (j \in \mathbb{N}^+ \setminus \{k, k+1\}), c_{i1} = c_{k+1, k+1} = 0, (i \in \mathbb{N}^+ \setminus \{k, k+1\})$.

With Proposition 4, we derive the following expressions:

$$
F_S (x) = H (x; \beta_1, \beta_2) + \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{\Gamma (i+j+1)} \times H (x; (k + i) \beta_1, (k + j) \beta_2),
$$

$$
E [S \times 1_{\{S > b\}}] = \xi (b; \beta_1, \beta_2) + \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{\Gamma (i+j+1)} \times \xi (b; (k + i) \beta_1, (k + j) \beta_2),
$$

and

$$
E [X_1 \times 1_{\{S > b\}}] = \xi_1 (x; \beta_1, \beta_2) + \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{\Gamma (i+j+1)} \times \xi_1 (x; (k + i) \beta_1, (k + j) \beta_2).
$$

4.2.4. Bivariate distribution defined with extensions of the FGM copula and exponential marginals

The FGM copula defined in (10) has led to various extensions. See e.g. Drouet-Mari and Kotz (2001) for a review on extensions of the FGM copula. In this paper, we only consider Huang and Kotz’s extension of the FGM copula. The other extensions can be treated similarly.

Huang and Kotz (1999) have investigated the behavior of two copulas within the FGM family in an effort to widen the range of attainable correlation coefficients. The extension proposed with the highest maximal correlation coefficient is defined by $C (u_1, u_2) = u_1 u_2 (1 + \theta (1 - u_1) (1 - u_2))$, $u_1, u_2 \in [0, 1]$, where $p > 0$. Here we only consider the case $p > 1$ since the only admissible value for $\theta$ is zero when $p < 1$. The admissible range for $\theta$ in such a case is $-1 \leq \theta \leq \left(\frac{p+1}{p}\right)^{-1}$. The expression for the joint pdf of $(X_1, X_2)$ is respectively given by

$$
f_{X_1, X_2} (x_1, x_2) = \beta_1 e^{-\beta_1 x_1} \beta_2 e^{-\beta_2 x_2} + \theta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \times (p + i) \beta_1 e^{-(p+i)\beta_1 x_1} \times (p + j) \beta_2 e^{-(p+j)\beta_2 x_2}.
$$

For a given $p > 1$, the restrictions on $\theta$ lead to the following bounds for the correlation coefficient

$$
-\frac{1}{[p(p+1)]^2} \leq \rho_F (X_1, X_2) \leq \frac{(p+1)^{p-3}}{p^2(p+1)^{p-1}},
$$

which means that the maximal positive correlation $\rho_F (X_1, X_2)_{\max} \approx 0.2600155$ is attained at $p \approx 1.042282$ and the maximal negative correlation is still $\rho_F (X_1, X_2)_{\min} = -\frac{1}{3}$ reached at $p = 1$. This provides a minor improvement over the standard FGM copula previously discussed. This bivariate distribution can be seen as a bivariate combination of exponentials with exponential marginals, i.e. $F_{X_1, X_2} \in \mathcal{C}^\epsilon (F_{X_1}, F_{X_2})$. Indeed, given (17), we have $m = 3, \gamma_1 = \beta_1, \lambda_1 = \beta_2, \gamma_i = (i - 1 + p) \beta_1 (i = 2, 3), \lambda_i = (j - 1 + p) \beta_2 (j = 2, 3), c_{i1} = 1, c_{i2} = c_{i1} = 0, c_{i3} = c_{i2} = (i \in \mathbb{N}^+ \setminus \{k, k+1\}), c_{k1} = c_{k+1, k+1} = \theta, c_{k, k+1} = -\theta, c_{i1} = c_{k+1, j} = 0, (j \in \mathbb{N}^+ \setminus \{k, k+1\}).$

With Proposition 4, we find the desired expressions for $F_S, E [S \times 1_{\{S > b\}}], E [X_1 \times 1_{\{S > b\}}]$.

4.2.5. Sarmanov’s bivariate exponential distribution

Sarmanov’s bivariate exponential distribution is a special case belonging to the family of distributions with arbitrary marginals introduced by Sarmanov (1966). Lee (1996) further studied this family introducing, among others, the bivariate exponential distribution defined by

$$
f_{X_1, X_2} (x_1, x_2) = \beta_1 \beta_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} \times \left\{1 + \theta \left( e^{-x_1} - \frac{\beta_1}{1 + \beta_1} \right) \left( e^{-x_2} - \frac{\beta_2}{1 + \beta_2} \right) \right\} + \left\{x_1 \times (k + i) \beta_1, (k + j) \beta_2 \right\},
$$

where

$$
\frac{-(1+\beta_1+\beta_2) \leq \theta \leq \frac{1+\beta_1+\beta_2}{\max(\beta_1, \beta_2)}},
$$

and

$$
\frac{1+\beta_1 + \beta_2 \leq \rho_F (X_1, X_2) \leq \frac{\beta_1 \beta_2}{(1+\beta_1)(1+\beta_2)} \leq \frac{1}{4},}
$$

where the bounds $-\frac{1}{4}$ and $\frac{1}{4}$ are achieved when $\beta_1 = \beta_2 = 1$. 

By simple manipulations, the joint pdf (18) of \((X_1, X_2)\) can be written as
\[
f_{X_1,X_2}(x_1, x_2) = \beta_1 \beta_2 e^{-(\beta_1 x_1 + \beta_2 x_2)} + \frac{\theta \beta_1 \beta_2}{(\beta_1 + 1)(\beta_2 + 1)} \times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} (\beta_1 + i) e^{-(\beta_1+i)x_1} (\beta_2 + j) e^{-(\beta_2+j)x_2}.
\]
(19)

Clearly, Sarmanov’s bivariate exponential distribution can be seen as a bivariate combination of exponentials with exponential marginals, i.e. \(F_{X_1,X_2} \in \Gamma^{\text{CE}}(F_{X_1}, F_{X_2})\). Indeed, given (19), the desired expressions can be found with Proposition 4 by letting \(m = 3\), \(\gamma_1 = \beta_1\), \(\lambda_1 = \beta_2\), \(\gamma_2 = \beta_1 + i - 2\) \((i = 2, 3)\), \(\lambda_j = \beta_2 + i - 2\) \((j = 2, 3)\), \(c_{1,1} = 1\), \(c_{1,2} = c_{1,3} = c_{2,1} = c_{3,1} = 0\), \(c_{2,2} = c_{3,2} = \frac{\theta \beta_1 \beta_2}{(\beta_1 + 1)(\beta_2 + 1)}\), and \(c_{2,3} = c_{3,3} = -\frac{\theta \beta_1 \beta_2}{(\beta_1 + 1)(\beta_2 + 1)}\).

5. Bivariate mixed Erlang distributions with exponential marginals

We continue our study of the aggregate sum and capital allocation within the class of bivariate mixed Erlang distributions with exponential marginals, denoted by \(\Gamma^{\text{ME}}(F_{X_1}, F_{X_2})\), which is a subset of the larger class of bivariate mixed Erlang distributions. In this section, we propose two methods to construct bivariate mixed Erlang distributions with exponential marginals. This first method lies on the compound geometric representation of an exponential distribution and leads notably to the well-known Moran–Downton bivariate exponential distribution. Such an approach also allows us to propose a generalized version of Moran–Downton’s bivariate exponential distribution. In the second method, we construct bivariate mixed Erlang distributions with exponential marginals from any absolutely continuous bivariate distribution with exponential marginals. In this section, we also briefly present the general class of bivariate mixed Erlang distributions, including Proposition 10 which provides the general expressions for \(F_5(x)\) \(\in [S \times 1_{\{\geq b\}}\) and \(E[X_1 \times 1_{\{\geq b\}}]\) for any bivariate mixed Erlang distribution and which is obviously applicable for any member of \(\Gamma^{\text{ME}}(F_{X_1}, F_{X_2})\).

5.1. Compound geometric representation of an exponential distribution

To construct bivariate mixed Erlang distributions with exponential marginals, one can use the compound geometric representation of an exponential distribution. Such a representation will frequently be used throughout this section.

An exponentially distributed rv \(X\) with parameter \(\beta\) and mgf given by \(\frac{\beta}{\beta - t}\) may be represented as the random sum \(X = \sum_{i=1}^{K} B_i\), where \(B_1, B_2, \ldots\) form a sequence of i.i.d. exponentially distributed rvs with mean \(\frac{\beta}{\beta}\) and also independent of a geometrically distributed rv \(K\) (defined on \(\mathbb{N}^+\)) with pmf, cdf and probability generating function (pgf) given by
\[
\Pr(K = k) = q(k; 1, \delta) = (1 - \delta)^{k-1}, \quad k \in \mathbb{N}^+, \quad 0 < \delta < 1,
\]
\[
\Pr(K \leq k) = Q(k; 1, \delta) = 1 - (1 - \delta)^k, \quad k \in \mathbb{N}^+, \quad 0 < \delta < 1,
\]
and \(\tilde{q}(s; 1, \delta) = E[s^K] = \frac{\delta}{1 - (1 - \delta)s}\). This implies that the pdf of \(X\) can also be written as
\[
f_X(x) = \sum_{k=1}^{\infty} q(k; 1, \delta) h\left(x; \frac{\beta}{\delta}\right).
\]
(20)

More generally, a rv \(X\) having an Erlang distribution with parameters \((n, \beta)\) can be represented as the same random sum, but in this case the rv \(K\) is defined on \((n, n + 1, \ldots)\) with pmf given by
\[
q(k; n, \delta) = \binom{k - 1}{k - n} (\delta)^n (1 - \delta)^{k-n}, \quad k = n, n + 1, \ldots,
\]
and pgf given by \(\tilde{q}(s; n, \delta) = \left(\frac{(1 - \delta)s}{1 - (1 - \delta)s}\right)^n\). Here, it leads to the pdf of \(X\) given by
\[
f_X(x) = \sum_{k=n}^{\infty} q(k; n, \delta) h\left(x; \frac{\beta}{\delta}\right).
\]
(22)

5.2. Bivariate mixed Erlang distributions

Lee and Lin (2012) propose a general class of bivariate mixed Erlang distributions, study its properties and examine interesting applications in actuarial science and quantitative risk management. We briefly recall its definition. Let a couple of rvs \((X_1, X_2)\) follow a bivariate mixed Erlang distribution whose joint pdf is defined as
\[
f_{X_1, X_2}(x_1, x_2) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} p_{k_1,k_2} h(x_1; k_1, \gamma) h(x_2; k_2, \gamma),
\]
(23)
where \(p_{k_1,k_2} \geq 0\) and \(\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} p_{k_1,k_2} = 1\). Note that the \(p_{k_1,k_2}\)’s can be seen as the values of the joint pmf of a pair of discrete rvs \((K_1, K_2)\), i.e. \(p_{k_1,k_2} = f_{K_1,K_2}(k_1, k_2)\), \(k_1, k_2 \in \mathbb{N}^+\). Given (23), the pair \((X_1, X_2)\) can be represented as \(X_1 = \sum_{k_1=1}^{\infty} B_{1,k_1}\) and \(X_2 = \sum_{k_2=1}^{\infty} B_{2,k_2}\), where \(\{B_{1,k_1}, k_1 \in \mathbb{N}^+\}\) and \(\{B_{2,k_2}, k_2 \in \mathbb{N}^+\}\) form sequences of i.i.d. rvs which are exponentially distributed with mean \(\frac{\beta}{\gamma}\). Moreover, \(\{B_{1,k_1}, k_1 \in \mathbb{N}^+\}\), \(\{B_{2,k_2}, k_2 \in \mathbb{N}^+\}\) and \((K_1, K_2)\) are independent.

The following proposition summarizes results for the aggregated sum which clearly follows a univariate mixed Erlang distribution and the expression needed for the capital allocation.

Proposition 9. Let \((X_1, X_2)\) follow a bivariate mixed Erlang distribution with joint pdf given by (23). Then, for \(S = X_1 + X_2\), we have
\[
F_S(x) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} p_{k_1,k_2} H(x; (k_1 + k_2), \gamma),
\]
(24)
\[
E[S \times 1_{\{\geq b\}}] = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} p_{k_1,k_2} \frac{k_1 + k_2}{\gamma} \tilde{H}\left(b; (k_1 + k_2) + 1, \gamma\right),
\]
and
\[
E[X_1 \times 1_{\{\geq b\}}] = \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} p_{k_1,k_2} \frac{k_1}{\gamma} \tilde{H}\left(b; (k_1 + k_2) + 1, \gamma\right).
\]

Proof. See Proposition 8 in Cossette et al. (2012). See also Lee and Lin (2012) for (24). □

An extension of the bivariate mixed Erlang distribution just defined can be derived by assuming two different rate parameters \(\gamma_1\) and \(\gamma_2\). This leads to the following joint pdf and joint mgf:
\[
f_{X_1,X_2}(x_1, x_2) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} p_{k_1,k_2} h(x_1; k_1, \gamma_1) h(x_2; k_2, \gamma_2)
\]
(25)
\[
M_{X_1,X_2}(r_1, r_2) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} p_{k_1,k_2} \left(\frac{\gamma_1}{\gamma_1 - r_1}\right)^{k_1} \left(\frac{\gamma_2}{\gamma_2 - r_2}\right)^{k_2},
\]
(26)
where \(p_{k_1,k_2} \geq 0\) and \(\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} p_{k_1,k_2} = 1\). We assume that \(0 < \gamma_1 \leq \gamma_2 < \infty\). Under this extension, the pair \((X_1, X_2)\) can still be represented as \(X_1 = \sum_{k_1=1}^{\infty} B_{1,k_1}\) and \(X_2 = \sum_{k_2=1}^{\infty} B_{2,k_2}\),
where $\{B_{1,i}, i \in \mathbb{N}^+\}$ form a sequence of i.i.d. rvs which are exponentially distributed with mean $\frac{1}{\gamma}$ and $\{B_{2,i}, i \in \mathbb{N}^+\}$ form a sequence of i.i.d. rvs which are exponentially distributed with mean $\frac{1}{\gamma}$ too. Also, $\{B_{1,i}, i \in \mathbb{N}^+\}$, $\{B_{2,i}, i \in \mathbb{N}^+\}$, and $(K_1, K_2)$ are independent.

We present the desired results for this proposed extension to the bivariate mixed Erlang distribution.

**Proposition 10.** Let $(X_1, X_2)$ follow a bivariate mixed Erlang distribution with joint pdf given by (25). Then, for $S = X_1 + X_2$ and for $\gamma_1 < \gamma_2$, we have

$$F_S(x) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k=1}^{\infty} p_{k_1,k_2} \sum_{k=1}^{\infty} q(k; k_1, \frac{\gamma_1}{\gamma_2}; k_2, \frac{\gamma_2}{\gamma_2}) \int H(x; (k + k_2) \cdot \gamma_2),$$

and

$$E[S \times 1_{\{S < b\}}] = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k=1}^{\infty} p_{k_1,k_2} \sum_{k=1}^{\infty} q(k; k_1, \frac{\gamma_1}{\gamma_2}; k_2, \frac{\gamma_2}{\gamma_2}) (k + k_2) \frac{1}{\gamma_2} \int \bar{H}(b; (k + k_2) + 1, \gamma_2),$$

where $q(k; r, \delta)$ is given in (21). For $\gamma_1 = \gamma_2$, see Proposition 9.

**Proof.** Combining (22) and (25) leads to

$$F_S(x) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k=1}^{\infty} p_{k_1,k_2} \sum_{k=1}^{\infty} q(k; k_1, \frac{\gamma_1}{\gamma_2}; k_2, \frac{\gamma_2}{\gamma_2}) \int h(x; (k + k_2) \cdot \gamma_2),$$

and

$$E[S \times 1_{\{S < b\}}] = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k=1}^{\infty} p_{k_1,k_2} \sum_{k=1}^{\infty} q(k; k_1, \frac{\gamma_1}{\gamma_2}; k_2, \frac{\gamma_2}{\gamma_2}) \int \frac{k}{\gamma_2} \int h(s; (k + k_2) + 1, \gamma_2),$$

from which the desired results follow.

**Remark 11.** Note that the result for $\text{TVaR}_R$ $(X_1; S)$ in Proposition 10 is also discussed in Theorem 1 of Willmot and Woo (2015). Indeed, the authors derived the contribution $\text{TVaR}_R$ $(X_1; S)$ in a multivariate setting with a representation based on only one scale parameter $\gamma$ due to Proposition 1 of Willmot and Woo (2014). This representation obviously allows to obtain expressions of the form provided in Proposition 9 for $F_S$ and $\text{TVaR}_R$ $(S)$.

### 5.3. Construction of bivariate mixed Erlang distributions with exponential marginals defined with bivariate geometric distributions

#### 5.3.1. Preliminaries

We propose a general method to generate a bivariate distribution with exponential marginals with parameters $\beta_1$ and $\beta_2$ within the class of bivariate mixed Erlang distributions using bivariate geometric distributions.

To this end, let $(X_1, X_2)$ follow a bivariate mixed Erlang distribution with joint pdf given by (25). We also assume that $(K_1, K_2)$ has a bivariate discrete distribution with geometric marginals whose joint pmf is denoted by $p_{k_1,k_2} = \Pr (K_1 = k_1, K_2 = k_2)$, for $k_1, k_2 \in \mathbb{N}^+$ and $\Pr (K_1 = k_1) = q(k_1; 1, 1 - \theta), k_1 \in \mathbb{N}^+, i = 1, 2$. Moreover, we suppose $\gamma_1 = \frac{1}{\beta_1}, i = 1, 2$. Given the compound geometric representation of an exponential distribution explained above, we find interestingly that the marginals of the bivariate mixed Erlang distribution of $(X_1, X_2)$ with cdf given in (25) are exponentials. Any bivariate geometric distribution leads to this result. This means that it is possible to generate a bivariate distribution with exponential marginals whose joint pdf is of the form given in (25) from a specific bivariate distribution for $(K_1, K_2)$.

By $\Gamma^{ME} (F_{X_1}, F_{X_2})$, we denote the resulting class of bivariate mixed Erlang distributions with exponential marginals. Clearly, $\Gamma^{ME} (F_{X_1}, F_{X_2})$ is a subset of the Fréchet class defined in Section 3 i.e. $\Gamma^{ME} (F_{X_1}, F_{X_2}) \subseteq \Gamma (F_{X_1}, F_{X_2})$.

A review on bivariate geometric distributions can be found in e.g. Kocherlakota and Kocherlakota (1992) and Johnson et al. (1997). Discrete bivariate distributions can be constructed with copulas as suggested e.g. in Joe (1997) and as illustrated e.g. in De-nuit et al. (2005), Cossette et al. (2002), and Genest et al. (2003), Trivedi and Zimmer (2005) and Marceau (2009). This type of structure permits the coupling of various marginals. Genest and Nešlehová (2007) provide an excellent review on copulas linking discrete distributions, where it is notably mentioned that dependence modeling with copulas as in (27) is a valid and attractive approach for constructing bivariate distributions. Many stochastic dependence properties of a copula are inherited by the bivariate model obtained in (27), such as the stochastic ordering relations which are preserved. If the joint cdf of $(K_1, K_2)$ is defined with a copula $C$, then we have

$$F_{K_1,K_2} (k_1, k_2) = C (F_{K_1} (k_1), F_{K_2} (k_2)) = C (Q (k_1; 1, 1 - \theta), Q (k_2; 1, 1 - \theta))$$

for $(k_1, k_2) \in \mathbb{N}^+ \times \mathbb{N}^+$. The joint pmf of $(K_1, K_2)$ is given by

$$p_{k_1,k_2} = \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} (-1)^{i_1+i_2} F_{K_1,k_2} (k_1 - i_1, k_2 - i_2),$$

for $(k_1, k_2) \in \mathbb{N}^+ \times \mathbb{N}^+$ and where $(k_1, k_2) \in \mathbb{N}^+ \times \mathbb{N}^+ = 0$ if $k_1 = 0$ or $k_2 = 0$. As mentioned in Nelsen (2006), $F_{K_1,k_2} (k_1, k_2) = C (F_{K_1} (k_1), F_{K_2} (k_2))$ is defined on the support of $(K_1, K_2)$. A given discrete bivariate distribution does not lead to a unique copula.

The expression for the covariance between $X_1$ and $X_2$ is $\text{Cov} (X_1, X_2) = \text{Cov} (K_1, K_2) \frac{\theta}{\beta_1 - \beta_2}$, which leads to

$$\rho_{X_1,X_2} = \text{Cov} (K_1, K_2) \frac{\theta}{\beta_1 - \beta_2}.$$
special case of the bivariate gamma distribution proposed by Kibble (1941) and discussed more generally by Krishnamoorthy and Parthasarathy (1951). A different form was suggested by Moran (1967). For further information on Moran–Downton’s bivariate exponential distribution, see e.g. Downton (1970), Kotz et al. (2004), or Iliopoulos (2003). This distribution has been applied in many fields, often related to environmental issues for which the distribution of $S$ is clearly of interest, as discussed in Nadarajah and Kotz (2006). Downton (1970) and Iliopoulos (2003) have mentioned that the pdf given in (29) can be expressed as

$$f_{X_1, X_2}(x_1, x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} q(k_1, k_2, \theta) f_{X_1, X_2}(x_1; k_1, \theta) f_{X_2, X_3}(x_2; k_2, \theta),$$

where the expression for $f_{X_1, X_2}(x_1; k_1, \theta)$ and $f_{X_2, X_3}(x_2; k_2, \theta)$ are exponential with rate parameters $k_1$ and $k_2$ respectively. Clearly, only positive dependence is possible. Also, independent and comonotonic rvs $X_1$ and $X_2$ are respectively obtained with $\theta = 0$ and $\theta \rightarrow 1$. The mgf of $(X_1, X_2)$ is given by

$$M_{X_1, X_2}(r_1, r_2) = \mathbb{E}\left( e^{r_1 X_1 + r_2 X_2} \right),$$

where $r_1$ and $r_2$ are non-negative. Using (28) with (32), we obtain

$$M_{X_1, X_2}(r_1, r_2) = \mathbb{E}\left( e^{r_1 X_1 + r_2 X_2} \right) \times (1 - \theta)^{k_1 + k_2 - 1},$$

for $k_1, k_2 \in \mathbb{N}^+$, which corresponds to a bivariate discrete distribution defined with the Fréchet upper bound copula and geometric marginals. Using (28) with (32), we obtain

$$p_{k_1, k_2} = \begin{cases} (1 - \theta)^{k_1 - 1} = q(k, 1 - \theta), & k_1 = k_2 = k \quad \text{if} \quad k_1 \neq k_2. \\ 0, & \end{cases}$$

Replacing (33) in (25) and (26), we obtain the pdf and the mgf (given respectively by (30) and (31)) of Moran–Downton’s bivariate exponential distribution.

We may refer to Proposition 10 to derive the expressions for $F_S$, $E\left[ S \times 1_{[S,b]} \right]$, and $E\left[ X_1 \times 1_{[S,b]} \right]$. However, in the following corollary, we obtain more concise expressions for $F_S$ and $E\left[ S \times 1_{[S,b]} \right]$, since it is shown that $S$ follows a generalized Erlang distribution.

**Corollary 12.** Let $(X_1, X_2)$ follow Moran–Downton’s bivariate exponential distribution with joint pdf given by (30). For $S = X_1 + X_2$, we obtain

$$M_S(r) = \mathbb{E}\left( e^{r S} \right) = \frac{(1 - \theta)}{\left(1 - \frac{1 - \theta}{k_1 + k_2} r\right)^2},$$

and

$$F_S(x) = H(x; \eta_1, \eta_2),$$

where

$$\eta_i = \frac{1}{2(1 - \theta)}\left(\beta_1 + \beta_2 + (-1)^i \sqrt{(\beta_1 + \beta_2)^2 - 4\beta_1 \beta_2 (1 - \theta)}\right),$$

for $i = 1, 2$. For all $0 \leq \theta \leq 0.5$.

**Proposition 13.** Let $(X_1, X_2)$ follow the generalized Moran–Downton bivariate exponential distribution with joint pdf given by (25) and $F_{X_1, X_2}$ defined by (39), where $0 \leq \theta \leq 0.5$. Then, the joint pmf of $(K_1, K_2)$ is

$$p_{k_1, k_2} = \begin{cases} \frac{1 - 2\theta}{\eta_1 (k_1 + k_2)}, & k_1 = 1, k_2 = 1 \\ (1 - \theta)^{k_1 - 1} = q(k_1, 1 - \theta), & k_1 = 1, k_2 = 2, \ldots \\ (1 - \theta)^{k_2 - 1} = q(k_2, 1 - \theta), & k_1 = 2, 3, \ldots, k_2 = 1 \\ 0, & \text{elsewhere} \end{cases}$$

Also, we have $E\left[ S \times 1_{[S,b]} \right] = \xi(b; \eta_1, \eta_2)$ and

$$E\left[ X_1 \times 1_{[S,b]} \right] = \sum_{k=1}^{\infty} q(k; 1, 1 - \theta) \sum_{k'=0}^{\infty} q\left(\beta_1 \beta_2 \right) \times \frac{1}{\beta_2} (b; k + j) + 1, \beta_2).$$

**Proof.** The expression in (34) for $M_S(r)$ follows from (31) and $M_S(r) = M_{X_1, X_2}(r, r)$. Then, from (34), we derive

$$M_S(r) = \frac{1}{\left(1 - \frac{1 - \theta}{\eta_1 r}\right)^2},$$

where the expression for $\eta_i$ is provided in (36). We deduce (35) from (38). The expression for (37) follows from Proposition 10 with (33).
and the covariance between \( K_1 \) and \( K_2 \) is

\[
\text{Cov}(K_1, K_2) = -\frac{\theta^2}{(1 - \theta)^2}
\]  \hspace{1cm} (41)

leading to the Pearson correlation coefficient \( \rho_p(X_1, X_2) = -\theta^2 \). The joint mgf of \( (X_1, X_2) \) is

\[
M_{X_1, X_2}(r_1, r_2) = \left( \frac{\beta_1}{\beta_2} - r_1 \right) \left( \frac{\beta_2}{\beta_1} - r_2 \right) + \left( \beta_1 - 1 \right) \frac{\beta_2}{\beta_1 - r_1} \frac{\beta_1}{\beta_2 - r_2}.
\]

For \( S = X_1 + X_2 \), we have

\[
F_S(x) = H\left(x; \frac{\beta_1}{1 - \theta}, \frac{\beta_2}{1 - \theta}\right) + H\left(x; \frac{\beta_1}{1 - \theta}, \frac{\beta_2}{1 - \theta}\right),
\]

\[
E[S \times 1_{S>0}] = \left(\frac{\beta_1}{1 - \theta}, \frac{\beta_2}{1 - \theta}\right) + \left(\frac{\beta_1}{1 - \theta}, \frac{\beta_2}{1 - \theta}\right),
\]

and

\[
E[X_1 \times 1_{S>0}] = \xi_1\left(\frac{\beta_1}{1 - \theta}, \frac{\beta_2}{1 - \theta}\right) + \xi_1\left(\frac{\beta_1}{1 - \theta}, \frac{\beta_2}{1 - \theta}\right).
\]

Proof. Using (28) with (39), we obtain (40). Using (40), we find (41). Then, with (40), we find the expression for \( M_{X_1, X_2}(r_1, r_2) \). Since \( M_S(T) = M_{X_1, X_2}(r_1, r_2) \), we can easily identify the expressions for \( F_S(x) \), \( E[S \times 1_{S>0}] \), and \( E[X_1 \times 1_{S>0}] \).

We derive from (39) an algorithm which allows the computation of the values of \( p_{k_1, k_2} \) when \( 0.5 < \theta < 1 \).

Algorithm 14. For \( 0.5 < \theta < 1 \), define \( k^* = \frac{\ln(0.5)}{\ln(\theta)} \) and \( k_2 = \left\lfloor \frac{\ln(1 - \theta)}{\ln(0.5)} \right\rfloor \). Then, we have \( p_{k_1, k_2} = 0 \) for all \( (k_1, k_2) \in \mathbb{N}^+ \times \mathbb{N}^+ \), except for the following pairs of \( (k_1, k_2) \):

1. for \( k_1 = k_2 = k^* : p_{k_1, k_2} = \begin{cases} 1 - 2\theta^{k^*}, & \text{if } 1 - \theta^{k^*-1}(1 - \theta) < 0 \\ 2\theta^{k^* - 1} - 1, & \text{if } 1 - \theta^{k^*-1}(1 - \theta) > 0 \\ \end{cases} \)

2. for \( k_1 = 1 \) and \( k_2 = k_2^{(k_1)}, k_2^{(k_1)} + 1, \ldots : p_{k_1, k_2} = p_{k_2, k_1} = \begin{cases} 1 - \theta^{k_1} - \theta^{k_2}, & \text{if } k_2 = k_2^{(k_1)} \\ (1 - \theta)\theta^{k_2 - 1}, & \text{if } k_2 > k_2^{(k_1)} \\ \end{cases} \)

3. for \( k_1 = 2, 3, \ldots \) and if \( k_2^{(k_1)} > k_1 \), for \( k_2 = k_2^{(k_1)}, \ldots, k_2^{(k_1)} - 1 \):

\[
p_{k_1, k_2} = p_{k_2, k_1} = \begin{cases} 1 - \theta^{k_1} - \theta^{k_2}, & \text{if } k_2 = k_2^{(k_1)} \\ (1 - \theta)\theta^{k_2 - 1}, & \text{if } k_2^{(k_1)} < k_2 < k_2^{(k_1)} - 1 \\ \theta^{k_1 - 1} + \theta^{k_2 - 1} - 1, & \text{if } k_2 = k_2^{(k_1)} - 1 \\ \end{cases} \)

The case for \( 0.5 < \theta \leq \frac{\sqrt{2}}{2} \) is developed in more detail in the following proposition.

Proposition 15. Let \((X_1, X_2)\) follow a generalized Moran–Downton bivariate exponential distribution with joint pdf given by (25) and \( F_{K_1, K_2} \) defined by (39), where \( 0.5 < \theta \leq \frac{\sqrt{2}}{2} \). Then, the joint pmf of \((K_1, K_2)\) is

\[
p_{k_1, k_2} = \begin{cases} 2\theta - 1, & k_1 = 2, k_2 = 2 \\ 1 - \theta - \theta^2, & k_1 = 1, k_2 = 2 \\ 1 - \theta - \theta^2, & k_1 = 2, k_2 = 1 \\ (1 - \theta)\theta^{k_2 - 1} = q(k_2; 1, 1 - \theta), & k_1 = 1, k_2 = 3, 4, \ldots \\ (1 - \theta)\theta^{k_1 - 1} = q(k_1; 1, 1 - \theta), & k_1 = 3, 4, \ldots, k_2 = 1 \\ 0, & \text{elsewhere}. \end{cases}
\]

The covariance of \((K_1, K_2)\) is \( \text{Cov}(K_1, K_2) = \frac{2\theta^3 - 6\theta^2 + 4\theta - 1}{(1 - \theta)^2} \), leading to a Pearson correlation coefficient given by \( \rho_p(X_1, X_2) = 2\theta^3 - 6\theta^2 + 4\theta - 1 \). The joint mgf of \((X_1, X_2)\) is

\[
M_{X_1, X_2}(r_1, r_2) = \left( \frac{\beta_1}{\beta_2} - r_1 \right) \left( \frac{\beta_2}{\beta_1} - r_2 \right) + \left( \beta_1 - 1 \right) \frac{\beta_2}{\beta_1 - r_1} \frac{\beta_1}{\beta_2 - r_2}.
\]

For \( S = X_1 + X_2 \), the expressions for \( F_S(x) \), \( E[S \times 1_{S>0}] \) and \( E[X_1 \times 1_{S>0}] \) can be determined easily from (43) using Proposition 10 with (21).

Proof. The joint pmf in (42) is derived using (28) with (39). Then, one finds (43) with (42).

5.4. Construction of bivariate mixed Erlang distributions with exponential marginals from bivariate exponential distributions

We propose a method to construct a bivariate mixed Erlang distribution with exponential marginals from any bivariate distribution with exponential marginals. The resulting bivariate mixed Erlang distribution is interesting in itself and it may also serve as an approximation of the initial bivariate distribution with exponential marginals. To this end, let us consider a pair of rv's \((X_1, X_2)\) which follows a bivariate distribution with exponential marginals with respective means \( \frac{1}{\beta_1} \) and \( \frac{1}{\beta_2} \) whose cdf \( F_{X_1, X_2} \) belongs to the Fréchet class \( \mathcal{F}(F_{X_1}, F_{X_2}) \) defined in Section 3.

In the proof of Theorem 2.1 (in Appendix A) of Lee and Lin (2012), we can find a method to construct an approximation \( F_{X_1, X_2}^{(i)}(x_1, x_2) \) of \( F_{X_1, X_2} \) defined in Section 3.

\[
F_{X_1, X_2}^{(i)}(x_1, x_2) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \alpha_{k_1, k_2} \beta_k \left( \frac{x_1}{k_1}, 1, \frac{x_2}{k_2}, 1, \frac{1}{\delta} \right) H_{k_1, k_2} \left( x_1; k_1, 1, \frac{1}{\delta} \right) \left( x_2; k_2, 1, \frac{1}{\delta} \right).
\]  \hspace{1cm} (44)
where
\[
\alpha_{k_1, k_2} = \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} F_{X_1, X_2} \left( (k_1 - i) \delta, (k_2 - i) \delta \right)
\]
\[
\times \int_{-\infty}^{\frac{-k_1 - 1}{\beta_1}} \frac{1}{\beta_1} \ln(1 - \delta) \, dy_1 \int_{-\infty}^{\frac{-k_2 - 1}{\beta_2}} \frac{1}{\beta_2} \ln(1 - \delta) \, dy_2,
\]
for any given \( \delta > 0 \). Clearly, the marginals of \((44)\) are univariate mixed Erlang, i.e. \(F_{X_i, X_j} \not\in \Gamma(F_{X_i}, F_{X_j})\). Moreover, the probabilities \(\alpha_{k_1, k_2}\) in \((45)\) contain both the information related to the marginals of \(F_{X_1, X_2}\) and the dependence structure of the joint distribution of \((X_1, X_2)\).

Here, we propose an alternative approach to construct an approximation of \(F_{X_1, X_2} \in \Gamma(F_{X_1}, F_{X_2})\), which is also a bivariate mixed Erlang distribution, but it is defined in such a way that its marginals are univariate exponentials rather than being univariate mixed Erlang as in \((44)\). To this end, for a fixed \(0 < \delta < 1\), we first apply Sklar’s representation theorem (see e.g. Denuit et al. 2005) to define the joint pmf of a pair of rvs \((K_1^{(\delta)}, K_2^{(\delta)})\) through the copula associated to \(F_{X_1, X_2}\) by
\[
f_{K_1^{(\delta)}, K_2^{(\delta)}}(k_1, k_2) = \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} F_{X_1, X_2} \left( F_{X_1}^{-1} \left( Q(k_1 - 1; 1, \delta) \right), \right.
\]
\[
\left. \times F_{X_2}^{-1} \left( Q(k_2 - 1; 1, \delta) \right) \right)
\]
\[
= \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} \ln(1 - \delta), \left. \frac{1}{\beta_1} \ln(1 - \delta), \frac{1}{\beta_2} \ln(1 - \delta) \right) 
\]
\[
\times \int_{X_1, X_2} \left( \left( -\frac{k_1 - i}{\beta_1} \right) \ln(1 - \delta), \left. \left( -\frac{k_2 - j}{\beta_2} \right) \ln(1 - \delta) \right) \right), \]
where \(F_{X_i}^{-1}(u) = \frac{1}{\beta_i} \ln(1 - u)\) is the inverse of \(F_{X_i}(x) = 1 - e^{-\beta_i x}\), for \(i = 1, 2\). Note that \((46)\) is equivalent to
\[
f_{K_1^{(\delta)}, K_2^{(\delta)}}(k_1, k_2) = \int_{-\infty}^{\frac{-k_1 - 1}{\beta_1}} \int_{-\infty}^{\frac{-k_2 - 1}{\beta_2}} \ln(1 - \delta) \, dy_1 \, dy_2.
\]

The bivariate distribution of the pair of rvs \((K_1^{(\delta)}, K_2^{(\delta)})\) is a bivariate geometric distribution with a dependence structure defined by the copula associated to \(F_{X_1, X_2}\). The dependence relation between \(K_1^{(\delta)}\) and \(K_2^{(\delta)}\) is hence inherited from the one between \(X_1\) and \(X_2\). The marginals of \(K_1^{(\delta)}\) and \(K_2^{(\delta)}\) are geometric with pmf \(q(k; 1, \delta), i = 1, 2\), see Marshall and Olkin (1995) for details. Let us now define the joint distribution of a pair of rvs \((X_1^{(\delta)}, X_2^{(\delta)})\) with joint cdf given by
\[
F_{X_1^{(\delta)}, X_2^{(\delta)}}(x_1, x_2) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} f_{K_1^{(\delta)}, K_2^{(\delta)}}(k_1, k_2)
\]
\[
\times H \left( x_1; k_1, \frac{\beta_1}{\delta} \right) H \left( x_2; k_2, \frac{\beta_2}{\delta} \right), \]
(48)

Clearly, based on the representation of an exponential distribution by a compound geometric distribution, \((X_1^{(\delta)}, X_2^{(\delta)})\) follows a bivariate mixed Erlang distribution with exponential marginals with means \(\frac{1}{\beta_1}\) and \(\frac{1}{\beta_2}\) respectively, i.e. from \((48)\), it is clear that \(F_{X_1^{(\delta)}, X_2^{(\delta)}} \in \Gamma^{\text{ME}}(F_{X_1}, F_{X_2}) \subset \Gamma(F_{X_1}, F_{X_2})\).

Compared to Lee and Lin (2012), we approximate here a bivariate distribution with exponential marginals by a bivariate mixed Erlang distribution that also has exponential marginals. Moreover, the approximation we propose is constructed in such a way that the dependence structure is separated from the marginals.

The class of bivariate distributions with joint pdf defined by \((48)\) and \((46)\) is dense in the Fréchet class \(\Gamma(F_{X_1}, F_{X_2})\) in the sense of weak convergence. Inspired from Theorem 2.9.1 of Tijms (1994) and similarly to Theorem 2.1 of Lee and Lin (2012), we have the following proposition.

**Proposition 16.** Let \(F_{X_1^{(\delta)}}, X_2^{(\delta)}\) be defined by \((48)\) and \(F_{X_1, X_2} \in \Gamma(F_{X_1}, F_{X_2})\). Then, we have
\[
limit_{\delta \to 0} \beta_1 \beta_2 = \Gamma_{X_1, X_2} (x_1, x_2) \to F_{X_1, X_2} (x_1, x_2),
\]
for any continuity points \((x_1, x_2)\) of \(F_{X_1, X_2} (x_1, x_2)\), i.e. \((X_1^{(\delta)}, X_2^{(\delta)})\) converges in distribution to \((X_1, X_2)\).

**Proof.** From \((48)\) and \((47)\), we have
\[
M_{X_1^{(\delta)}, X_2^{(\delta)}}(r_1, r_2) = \int_0^\infty \int_0^\infty F_{X_1, X_2} (y_1, y_2) \left( 1 - \frac{\delta}{\beta_1} \right) \left( 1 - \frac{\delta}{\beta_2} \right) \, dy_1 \, dy_2.
\]

where \(|u|\) is the ceiling function, which gives the smallest integer greater than \(u\). For \(0 \leq \frac{\delta}{\beta_i} r_i < 1\), we have
\[
limit_{\delta \to 0} \left( 1 - \frac{\delta}{\beta_i} r_i \right)^{-\frac{1}{\gamma_i}} = e^{\gamma_i},
\]
for \(i = 1, 2\). Then, by the Dominated Convergence Theorem, we have
\[
limit_{\delta \to 0} \left( 1 - \frac{\delta}{\beta_i} r_i \right)^{-\frac{1}{\gamma_i}} \to e^{\gamma_i},
\]
(49)

By Levy’s Continuity Theorem, \((49)\) implies that \(F_{X_1^{(\delta)}, X_2^{(\delta)}} (x_1, x_2)\) converges to \(F_{X_1, X_2} (x_1, x_2)\) for any continuity points \((x_1, x_2)\) of \(F_{X_1, X_2} (x_1, x_2)\).

In order to illustrate an interesting aspect of this new distribution, let us consider a couple of rvs \((X_1, X_2)\) which follows a bivariate distribution with exponential marginals with cdf \(F_{X_1, X_2} \in \Gamma(F_{X_1}, F_{X_2})\). For \(S = X_1 + X_2\) and given \(F_{X_1, X_2}\), we assume that
no analytic expressions for $F_s$, $E\left[S \times 1_{\{S > b\}}\right]$, and $E\left[X_1 \times 1_{\{S > b\}}\right]$ can be found. For a given $0 < \delta < 1$, we can define a pair of rv's $(X^{(\delta)}, Y^{(\delta)})$ which follows a bivariate mixed Erlang distribution with $F_1^{(\delta)}, F_2^{(\delta)}$ defined as in (48) and with the same exponential marginals as $F_{X_1, X_2}$ (i.e. $F_1^{(\delta)} x_2^{(\delta)} \in \Gamma\{F_{X_1}, F_{X_2}\}$). For $S^{(\delta)} = X^{(\delta)} + Y^{(\delta)}$, it means that, by Proposition 16, $S^{(\delta)}$ also converges in distribution to $S$ as $\delta \to 0$. Therefore, we can use the values of $F_1^{(\delta)}$, $E\left[S^{(\delta)} \times 1_{\{S^{(\delta)} > b\}}\right]$, and $E\left[X^{(\delta)}_1 \times 1_{\{S^{(\delta)} > b\}}\right]$, which are computed with Proposition 10, to approximate the corresponding values of $F_s$, $E\left[S \times 1_{\{S > b\}}\right]$, and $E\left[X_1 \times 1_{\{S > b\}}\right]$.

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