Explicit analytic ruin probabilities for bounded claims

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Abstract

We demonstrate how explicit analytic ruin probabilities can be obtained for claimsize distributions concentrated on a compact interval and constructed arbitrarily from polynomials, exponential functions $e^{ax}$ ($a \in \mathbb{R}$) and trigonometric functions $\sin(bx)$ ($x \in \mathbb{R}$), $\cos(cx)$ ($x \in \mathbb{R}$), by a finite number of additions and multiplications. The method applies also in case of unbounded claims, and then its execution is much simpler than for bounded claims.

Keywords: Classical risk model; Convolution; Renewal equation; Function of bounded variation

1. Various functions / factorial transforms

1.1. Definitions

All considered functions are real (until 4.9). They are defined everywhere if nothing else is specified. By everywhere we mean on $\mathbb{R}_+ = [0, \infty[$. Some functions (factorial transforms), are defined near 0 only. By near 0 we mean on some interval $[0, \epsilon]$. By interval we always mean a non-void interval, not reduced to a single point.

A distribution function $F$ is a positive (weak sense) increasing (weak sense) right-continuous function. The distribution function $F$ is identified with the distribution it defines on the Borel sets of $\mathbb{R}_+$. The distribution $F$ attaches the mass $F(0)$ at the origin. The distribution $F$ is a probability distribution, or briefly, a probability, if $F(\infty) = 1$.

A BV-function is a right-continuous function of bounded variation on compact intervals. The function $F$ is a BV-function iff it is the difference of two distribution functions.

A B-function is a Borel function bounded on bounded sets.

These definitions can be restricted to some interval $[0, b]$. For instance, $F$ is a probability distribution on $[0, b]$ means that the restriction of $F$ to $[0, b]$ is positive, increasing, right-continuous and $F(b) = 1$. Then nothing is specified about $F$ outside $[0, b]$.

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The factorial transform $T$ and its inverse $T^{-1}$ are defined on formal (convergent or not) power series around 0, by

$$T\left( \sum a_k x^k \right) = \sum k! a_k x^k, \quad T^{-1}\left( \sum b_k x^k \right) = \sum \frac{1}{k!} b_k x^k.$$ 

Any function is identified with its power series expansion around 0, on the domain of existence and convergence.

An entire function is a function with everywhere convergent power series expansion. Entire functions are BV-functions (because their derivatives are B-functions). A very regular function (this is not a standard definition) is an entire function $f$ such that $Tf$ converges near 0. We say that $f$ is very regular on $[0, b]$, $[0, b[$, if a very regular function $f^0$ exists on $\mathbb{R}_+$ such that $f$ and $f^0$ coincide on $[0, b]$, $[0, b[$ respectively.

Factorial transforms of very regular functions serve only to recover the original functions. It is useless to indicate their precise radius of convergence.

But in some proofs, this radius is needed. We recall that the radius of convergence $r$ of the power series $\sum a_k x^k$ satisfies

$$\frac{1}{r} = \lim \sup_{k} |a_k|^{1/k} \in [0, \infty], \quad (1)$$

and that

$$\lim \sup_{k} b_k < c$$

implies the existence of $k_0$ such that $b_k \leq c \ (k \geq k_0)$.

1.2. Theorems

**Theorem 1.** If $f$ is a power series converging near 0, then $T^{-1}f$ is an entire function.

**Proof.** Let $r, R$ be the radius of convergence of

$$f(x) = \sum a_k x^k \quad \text{and} \quad T^{-1}f(x) = \sum \frac{1}{k!} a_k x^k$$

respectively. Then $r$ satisfies (1) and $R$ satisfies

$$\frac{1}{R} = \lim \sup_{k} \left| \frac{a_k}{k!} \right|^{1/k}.$$ 

We assume $r > 0, r^{-1} < \infty$. By Stirling $\lim(k!)^{1/k} = \infty$. Hence $R^{-1} = 0$, i.e. $R = \infty$.

**Theorem 2.** Let $f$ be a very regular function and $c$ a positive constant. Then $f(c + x)$ is a very regular function of $x$.

**Proof.** Let $f(x) = \sum a_k x^k$. The function $f$ being entire, $f(c + x)$ can be expanded as follows:

$$f(c + x) = \sum a_k (c + x)^k = \sum \sum a_k \binom{k}{j} c^{k-j} x^j,$$

$$= \sum \sum a_k \binom{k}{j} x^j c^{k-j} = \sum \left( \sum a_{k+j} \binom{k+j}{j} c^k \right) x^j.$$ 

Then

$$Tf(c + x) = \sum_{j \geq 0} \left( \sum_{k \geq j} a_{k+j} \frac{(k+j)!}{k!} c^k \right) x^j.$$
The radius of convergence $r$ of $Tf(x)$ is strictly positive. Hence
\[
\lim \sup (k! |a_k|)^{1/k} = \frac{1}{r} < \infty.
\]

Let $b < 1/r$ be fixed. For some $k_0$ and then all $k \geq k_0$:
\[
(k! |a_k|)^{1/k} \leq b, \quad k! |a_k| \leq b^k.
\]
This implies that for some $M$ and then all $k \geq 0, k! |a_k| \leq Mb^k$. Then, for $x < 1/b$
\[
\sum_j \sum_k |a_{k+j}| \frac{(k+j)!}{k!} c^k x^j \leq \sum_j \sum_k Mb^{k+j} \frac{1}{k!} c^k x^j
\]
\[
= M \sum_k \frac{1}{k!} (bc)^k \sum_j (bx)^j = M b^c c \cdot \frac{1}{1 - bx} < \infty.
\]
This means that the radius of convergence of $Tf(c + x)$ equals $1/b$ at least. Hence, it is strictly positive, and $T(c + x)$ is a very regular function of $x$.

1.3. Remarks
(a) The class of very regular functions is closed under several operations: translation (1.3), convolution (next 2.2), derivation (obvious), formation of linear combinations (obvious), usual product (not proved or used in this note), etc. The usual functions polynomials, $e^x$, sin $x$, cosh $x$, ... are very regular. Examples of entire, not very regular functions are
\[
\exp(x^2), \quad \sum \frac{1}{(k!)^{1/2}} x^k.
\]
(b) In the next theorem, we consider a function $f(x, s)$ of two variables $x$ and $s$. The current variable remains $x$, and the preceding definitions apply to the functions of $x$, for fixed $s$. The theorem can be stated briefly as follows: If $f(x, s)$ is very regular for $s$ near $s_0$ and $Tf(x, s)$ is an analytic function of $(x, s)$ for $x$ near 0 and $s$ near $s_0$, then $Tf(x, s)$ can be differentiated in $s$, any number of times, and the differentiations may be taken under the $T$ operator.

1.4. Theorem and corollary

Theorem 3. Let $f(x, s)$ be a very regular function of $x$ for each $s \in I = [s_0 - h, s_0 + h]$. Suppose that
\[
Tf(x, s) = \sum_{ij} a_{ij} x^i (s - s_0)^j \quad ((x, s) \in [0, r[xI])
\]
and that the double series converges absolutely on the indicated domain $[0, r[xI]$. Then
\[
\left( \frac{\partial}{\partial s} \right)^k f(x, s) \quad (k \geq 1)
\]
exists and it is very regular function of $x$ for each $s \in I$, and
\[
T \left( \frac{\partial}{\partial s} \right)^k f(x, s) = \left( \frac{\partial}{\partial s} \right)^k Tf(x, s) \quad ((x, s) \in [0, r[xI].
\]
Proof. In series or double series with positive terms, and in absolutely convergent series or double series, the terms may be re-arranged in any order. In power series, or double power series like (2), derivatives or partial derivatives may be taken termwise, any number of times, and the absolute convergence subsists in the differentiated series. The proof is based on these properties. Let \( g(x, s) \) be defined on \([0, r[xI]\) by

\[
g(x, s) = \sum_{ij} |a_{ij}| x^i |s - s_0|^j
\]

\[
= \sum_i \left( \sum_j |a_{ij}| |s - s_0|^j \right) x^i.
\]

Then \( g(x, s) \) is finite on \([0, r[xI]\) because (2) converges absolutely. By Theorem 1, for fixed \( s \in I \),

\[
T^{-1} g(x, s) = \sum_i \left( \sum_j |a_{ij}| |s - s_0|^j \right) \frac{1}{i!} x^i
\]

\[
= \sum_{ij} |a_{ij}| \frac{1}{i!} x^i |s - s_0|^j
\]

is finite, for all \( x \in \mathbb{R}_+ \). This means that the double series

\[
\sum_{ij} a_{ij} \frac{1}{i!} x^i (s - s_0)^j
\]

(3)

converges absolutely on \( \mathbb{R}_+ x I \).

From (2),

\[
Tf(x, s) = \sum_i \left( \sum_j a_{ij} (s - s_0)^j \right) x^i \text{ on } [0, r[xI],
\]

and then

\[
f(x, s) = T^{-1} Tf(x, s) = \sum_i \left( \sum_j a_{ij} (s - s_0)^j \right) \frac{1}{i!} x^i
\]

\[
= \sum_{ij} a_{ij} \frac{1}{i!} x^i (s - s_0)^j \text{ on } \mathbb{R}_+ x I,
\]

where the last double series is the absolutely convergent double series (3).

Then, on \( \mathbb{R}_+ x I \)

\[
\left( \frac{\partial}{\partial s} \right)^k f(x, s) = \sum_{ij} a_{ij} \frac{x^i}{i!} j^{[k]} (s - s_0)^{j-k}
\]

\[
= \sum_i \left( \sum_j a_{ij} \frac{1}{i!} j^{[k]} (s - s_0)^{j-k} \right) x^i,
\]

where \( j^{[k]} = j(j-1)\ldots (j-k+1) \). Then, formally,

\[
T \left( \frac{\partial}{\partial s} \right)^k f(x, s) = \sum_i \left( \sum_j a_{ij} j^{[k]} (s - s_0)^{j-k} \right) x^i.
\]
From (2), we have, on \([0, r[xI]\),
\[
\left( \frac{\partial}{\partial s} \right)^k T f(x, s) = \sum_{ij} a_{ij} x^i j^k (s - s_0)^{j-k} \\
= \sum_i \left( \sum_j a_{ij} j^k (s - s_0)^{j-k} \right) x^i.
\]
The latter series converges on \([0, r[xI]\) and it is the series in the last member of (4). Hence, for fixed \(s \in I, (\partial/\partial s)^k f(x, s)\) is a very regular function of \(x\), and
\[
T \left( \frac{\partial}{\partial s} \right)^k f(x, s) = \left( \frac{\partial}{\partial s} \right)^k T f(x, s) \quad \text{on} \quad [0, r[xI].
\]

**Corollary 1.** We assume the following: \(g(x)\) has a power series expansion convergent near 0, \(|s_0 g(0)| < 1, f(x, s)\) is a very regular function of \(x\) for \(s\) near \(s_0\),
\[
Z f(x, s) = (1 - sg(x))^{-1}
\]
for \(x\) near 0 and \(s\) near \(s_0\).
Then \((\partial/\partial s)^k f(x, s) (k \geq 1)\) is a very regular function of \(x\) for \(s\) near \(s_0\) and
\[
T \left( \frac{\partial}{\partial s} \right)^k f(x, s) = \left( \frac{\partial}{\partial s} \right)^k T f(x, s) = k! (g(x))^k (1 - sg(x))^{-k-1}
\]
for \(x\) near 0 and \(s\) near \(s_0\).

2. **Convolutions**

**Definition.** The *convolution* of the B-function \(f\) with the BV-function \(F\) is the B-function \(f * F\) defined everywhere by
\[
(f * F)(x) = \int_{0-}^{x+} f(x - y) \, dF(y)
\]
\[
= \int_{0+}^{x-} f(x - y) \, dF(y) + f(x) F(0) + f(0) (F(x) - F(x-)).
\]

**Theorem 4.** If \(f\) and \(F\) are very regular, then \(f * F\) is very regular and \(T(f * F) = T f . T F\) near 0.

**Proof.** Let
\[
f(x) = \sum a_i x^i, \quad F(x) = \sum b_j x^j
\]
be very regular. Let us assume first that \(a_i \geq 0, b_j \geq 0\). Then
\[
(f * F)(x) = \int_0^x f(x - y) F'(y) \, dy + f(x) F(0)
\]
\[
= \int_0^x \sum_{i \geq 0} a_i (x - y)^i \sum_{j \geq 1} b_j y^{j-1} \, dy + f(x) F(0)
\]
\[
= \sum_{i \geq 0} \sum_{j \geq 1} a_i b_j \int_0^x (x - y)^i y^{j-1} \, dy + f(x) F(0)
\]
\[ T(\mathbf{a}, \mathbf{b})(x) = \sum_{i \geq 0} \sum_{j \geq 0} a_i b_j \frac{i! j!}{(i+j)!} x^{i+j} + \sum_{i \geq 0} a_i b_0 x^i \]
\[ = \sum_{i \geq 0} \sum_{j \geq 0} a_i b_j \frac{i! j!}{(i+j)!} x^{i+j}, \quad (5) \]

where the transformations are allowed because all terms are positive in the double series (5) (or by monotone convergence).

\[ T(\mathbf{f} \ast \mathbf{F})(x) = T \left( \sum_{k \geq 0} \left( \sum_{i \leq k} a_i b_{k-i} \frac{1}{k!} i!(k-i)! \right) x^k \right) \]
\[ = \sum_{k \geq 0} \left( \sum_{i \leq k} a_i b_{k-i} i!(k-i)! \right) x^k = \sum_{i \geq 0} \sum_{j \geq 0} a_i b_j x^{i+j} \]
\[ = \left( \sum_{i \geq 0} a_i i! x^i \right) \left( \sum_{j \geq 0} b_j j! x^j \right) = T\mathbf{f}(x).T\mathbf{F}(x), \quad (6) \]

everywhere (but with \( \infty = \infty \) not excluded).

Let us now drop the assumption \( a_i \geq 0, b_j \geq 0 \). Let

\[ f_0(x) = \sum |a_i| x^i, \quad F_0(x) = \sum |b_j| x^j, \]

then

\[ (f_0 \ast F_0)(x) = \sum_i \sum_j |a_i| |b_j| \frac{i! j!}{(i+j)!} x^{i+j}. \]

The first member is finite everywhere. This proves that the double series (5) converges absolutely everywhere. Its terms can be re-arranged in any order, and the transformations leading to (5) are justified now by dominated convergence. Hence: \( (\mathbf{f} \ast \mathbf{F})(x) \) is an entire function, with absolutely convergent double series expansion (5) everywhere.

The series \( T\mathbf{f} \) and \( T\mathbf{F} \) converge on some interval \([0, r[\), and then they converge absolutely on \([0, r[, i.e. \( Tf_0 \) and \( TF_0 \) converge on \([0, r[\). Hence, for \( x \in [0, r[\),

\[ \infty > Tf_0(x).TF_0(x) = \sum_i \sum_j |a_i| i! |b_j| j! x^{i+j} \]
\[ = \sum_{k \geq 0} \left( \sum_{i \leq k} a_i b_{k-i} i!(k-i)! \right) x^k. \]
\[ (7) \]

From (7) results that the double series

\[ \sum_i \sum_j a_i i! b_j j! x^{i+j} \]
\[ (9) \]

converges absolutely on \([0, r[\), and that its terms can be re-arranged in any order on \([0, r[\). We have

\[ T(\mathbf{f} \ast \mathbf{F})(x) = T \sum_k \left( \sum_{i \leq k} a_i b_{k-i} \frac{1}{k!} i!(k-i)! \right) x^k \]
\[ = \sum_k \left( \sum_{i \leq k} a_i b_{k-i} i!(k-i)! \right) x^k. \]
The comparison with (8) shows that the expansion of $T(f \ast F)(x)$ converges on $[0, r]$. Hence $f \ast F$ is very regular. From the absolute convergence of (9) results that the transformations (6) are allowed on $[0, r]$. Hence $T(f \ast F) = Tf.TF$ near 0.

**Theorem 5.** If $f$ and $F$ are entire functions, then $f \ast F$ is an entire function.

**Proof.** The conclusion that $f \ast F$ is an entire function, in the preceding proof, is based only on the assumption that $f$ and $F$ are entire functions. It does not use the very regularity assumption.

3. **Renewal equations**

3.1. **Definitions**

A renewal equation for $f$ is an equation $f = g + f \ast G$, where $g$ is a B-function and $G$ a BV-function. A solution of the equation, is a B-function satisfying it. We say that $f$ is a solution on the compact interval $[0, b]$ of the renewal $f = g + f \ast G$ if $f$ is a function defined at least on $[0, b]$, bounded and Borel on $[0, b]$, and satisfying the equation on $[0, b]$, i.e. such that

$$f(x) = g(x) + (f \ast G)(x) \quad (0 \leq x \leq b).$$

We notice that the value $(f \ast G)(x)$ of the convolution $f \ast G$, at the point $x$, only depends on the restriction of $f$ and $G$ to the interval $[0, x]$. Hence, $(f \ast G)(x)$ in (10), is defined for all $x \in [0, b]$, even if $f$ is not defined outside $[0, b]$.

3.2. **Theorems**

**Theorem 6.** We assume the following about the renewal equation $f = g + f \ast G$: $G$ is a distribution function on the compact interval $[0, b]$, $G(b) < 1$, $U$ is defined on $[0, b]$ by

$$U = \sum_{k \geq 0} G^{*k} = 1 + \sum_{k \geq 1} G^{*k}.$$  

Then $U$ is a distribution function on $[0, b]$, $U(b) < \infty$ and $g \ast U$ is a solution on $[0, b]$ of the given renewal equation. Any other solution on $[0, b]$ of the equation, coincides with $g \ast U$ on $[0, b]$.

Hence, on $[0, b]$ the solution of the renewal equation $f = g + f \ast G$ exists, is unique, and is determined by the restriction of $g$ and $G$ to $[0, b]$.

**Proof.** We first prove the unicity. For any distributions $F$, $G$ on $[0, b]$:

$$(F \ast G)(b) \leq F(b).G(b).$$

This implies

$$G^{*k}(b) \leq (G(b))^k \quad (k \geq 0).$$

Let $f$, $f_0$ be solutions on $[0, b]$ of the given renewal equation. Then

$$f = g + f \ast G \quad [0, b], \quad f_0 = g + f_0 \ast G \quad [0, b].$$

By subtraction

$$f - f_0 = (f - f_0) \ast G \quad [0, b].$$
Iterating
\[ f - f_0 = (f - f_0) \ast G^n \text{ on } [0, b]. \]
Let \( B \) be an upper bound for \(|f|\) and \(|f_0|\) on \([0, b]\). For any \( x \in [0, b] \):
\[
|f(x) - f_0(x)| \leq \int_0^x |f(x - y) - f_0(x - y)| \ dG^n(y)
\leq \int_0^x 2B \ dG^n(y) = 2BG^n(x) \leq 2BG^n(b)
\leq 2B(G(b))^n \to 0 \text{ for } n \to \infty.
\]
Hence, \( f = f_0 \) on \([0, b]\).

In order to prove that \( g \ast U \) is solution on \([0, b]\) of the given renewal equation, we apply the following classical result with \( q = 1 \) and \( F \) defined by the conditions
\[ F = G \text{ on } [0, b], \quad F = 1 \text{ on } [b + 1, \infty[, \quad F \text{ linear continuous on } [b, b + 1]. \]

Then \( V(b) < \infty \), \( U = V \) on \([0, b]\), \( g \ast U = g \ast V \) on \([0, b]\). \( g \ast V \) being a solution of \( f = g + f \ast F \), it follows that \( g \ast U \) is a solution on \([0, b]\) of \( f = g + f \ast G \).

**Theorem 7.** The renewal equation \( f = g + qf \ast F \) where \( 0 < q < 1 \) and \( F \) is a probability distribution, has the unique solution \( f = g \ast V \), where
\[
V = \sum_{k \geq 0} q^k F^k
\]
is a distribution.

**Proof.** This is a classical result. See Feller (1966).

**Theorem 8.** The renewal equation \( f = g + f \ast G \), where \( g \) and \( G \) are very regular and \( |G(0)| < 1 \) has a unique very regular solution \( f \).

**Proof.** Let
\[
g(x) = \sum b_k x^k, \quad G(x) = \sum c_k x^k. \tag{12}
\]
We first prove unicity. Let
\[
f(x) = \sum a_k x^k \tag{13}
\]
be a very regular solution of \( f = g + f \ast G \). Taking factorial transforms, we have \( Tf = Tg + Tf.TG \), where \( Tf, Tg, TG \) converge near 0. An identification of coefficients leads to the relations
\[
a_0 = \frac{b_0}{1 - c_0}, \quad a_k = \frac{1}{1 - c_0} \left( b_k + \sum_{i \leq k - 1} a_i c_{k - i} \frac{1}{k!} i!(k - i) \right) \tag{14}
\]
determining, successively, \( a_0, a_1, \ldots \). Hence, the regular solution must be unique, if it exists.

In order to prove the existence, we consider the expansion
\[
\frac{Tg(x)}{1 - TG(x)} = Tg(x) \cdot \sum_i (TG(x))^i = \sum_k d_k x^k
\]
converging near 0, and we define \( f \) by
\[
f(x) = T^{-1} \left( \frac{T_g(x)}{1 - TG(x)} \right) = \sum_k \frac{1}{k!} d_k x^k.
\]
Then \( f \) is an entire function by Theorem 1, and
\[
Tf(x) = \sum_k d_k x^k
\]
converges near 0. Hence \( f \) is very regular. We have near 0
\[
Tf(x) = \frac{T_g(x)}{1 - TG(x)}, \quad Tf(x)(1 - TG(x)) = Tg(x),
\]
\[
Tf(x) = Tg(x) + Tf(x).TG(x), \quad Tf = T(g + f \ast G).
\]
Hence \( f = g + f \ast G \) and \( f \) is the very regular solution of the given renewal equation.

**Remark.** The relations (14) may be useful for the determination of the first terms of the development (13) of \( f \) and for the study of the behaviour of \( f \) near 0.

4. Classical risk theory

4.1. The renewal equation

The renewal equation of classical risk theory is
\[
U(p + qU \ast G)
\]
where \( G \) is the probability distribution defined by
\[
G(x) = \frac{1}{\mu} \int_0^x (1 - F(y)) \, dy,
\]
where \( F \) is the distribution function of the partial claims, with first moment \( \mu \in ]0, \infty[ \).

In (15), \( p = \eta (1 + \eta)^{-1}, q = (1 + \eta)^{-1}, 0 < \eta = \text{security loading} \). \( U(x) \) is the probability of non-ruin corresponding to the initial reserve \( x \). From (16) results that \( G \) is continuous, with \( G(0) = 0 \).

The function \( U \) is a probability distribution function, continuous on \( \mathbb{R}_+ \), but with \( U(0) = p > 0 \). Hence, \( U \), as a distribution, has an atom at the origin (and cannot be considered a continuous distribution).

We say that a distribution \( H \) is concentrated on \( [0, b] \) if it places no mass outside \( [0, b] \), i.e. iff \( H(x) = H(b) (x \geq b) \). The probability distribution \( H \) is concentrated on \( [0, b] \) iff \( H(x) = 1(x \geq b) \).

4.2. Associated very regular renewal equation

Let us assume here that \( G \) is very regular on some compact interval \([0, b]\). Let \( G^0 \) be the very regular function such that \( G = G^0 \) on \([0, b]\) and let us then consider the very regular renewal equation
\[
U^0 = p + qU^0 \ast G^0
\]
associated to the original renewal equation \( U = p + qU \ast G \). On \([0, b]\), \( G \) and \( G^0 \) coincide. Hence \( U = U^0 \) on \([0, b]\) by Theorem 6 (where the condition \( G(b) < 1 \) is always satisfied, because it becomes here \( qG(b) < 1 \)).
The study of (17) may be much simpler than that of (15) and provides information on $U$ near 0.

4.3. $U$-curves with prescribed shape near 0

(a) Schmitter's problem is the determination of the extremal claims size distribution $F$ concentrated on a fixed interval $[0, b]$ and with fixed first and second order moments, leading, for a fixed initial risk reserve $u$, to the largest probability of ruin. In the proof of the fact that the extremal $F$ is necessarily atomic, before $u$, we wondered whether the function $U*U$ could have linear pieces. The answer is affirmative, as we show in (d).

(b) Let us indicate how to obtain a prescribed shape for $U$ near 0. Let the very regular expansion

$$U^0(x) = p(1 + a_1 x + a_2 x^2 + \ldots)$$

be given, and assume

$$a_1 > 0, \quad \mu a_1 < q, \quad 2a_2 < a_1^2.$$ 

Then there exists an interval $J=[0, b]$ and a claims size distribution $F$, with first moment $\mu$, and corresponding $G$ and $U$ such that $U = U^0$ on $J$.

Indeed, let $U_0(x) = a_1 x + a_2 x^2 + \ldots$. Then $U^0 = p(1 + U_0)$. Using this in the renewal equation

$$U^0 = p + qU^0 * G^0, \quad G^0$$

has to satisfy the equation

$$U_0 = q(1 + U_0) * G^0.$$ 

Taking the factorial transform, we must have, near 0

$$TG^0 = \frac{1}{q}TU_0(1 + TU_0)^{-1} = \frac{1}{q}TU_0(1 - TU_0 + (TU_0)^2 - \ldots).$$

Explicitly

$$TG^0(x) = \frac{1}{q}(a_1 x - (a_1^2 - 2a_2)x^2 + \ldots)$$

for the first (only relevant) terms. Then, near 0, it is enough to take

$$F(x) = 1 - \mu G^0(x) = \left(1 - \frac{\mu}{a_1}\right) + \frac{\mu}{q}(a_1^2 - 2a_2)x + \ldots.$$  

If $J$ is small enough, this $F$ is a distribution on $J$, and can be extended as a probability distribution on $\mathbb{R}_+$ with first moment $\mu$. Indeed, taking $J$ small enough, the impact of the mass distribution on $J$ becomes negligible for the first moment constraint.

We notice that $TG^0$ defined by (18) converges near 0. Hence, $G^0$ is an entire function (Theorem 1) and then the derivative $G^0'$ is also en entire function. Hence, the last member of (19) is an entire function. It converges everywhere, in particular on $J$.

(c) As a first illustration, we want $U$ linear near 0. Applying (a) and giving particular values to the parameters, we obtain:

For $F$ defined by

$$F(x) = \begin{cases} 1 - \frac{1}{2}e^{-x} & (x < -\log p) \\ 1 & (x \geq -\log p) \end{cases}$$
we have
\[ \mu = \frac{q}{2}, \quad G(x) = \frac{1}{q}(1 - e^{-x}) \quad (x < -\log p), \quad U(x) = p(1 + x) \quad (x < -\log p). \]

(d) Next we want \( U \ast U \) linear near 0, say \( (U \ast U^0)(x) = p^2(1 + ax) \). Then,
\[ (TU^0(x))^2 = p^2(1 + ax), \]
\[ TU^0(x) = p(1 + ax)^{1/2} = p\left(1 + \frac{1}{2}ax - \frac{1}{8}a^2x^2 \ldots\right), \]
\[ U^0(x) = p\left(1 + \frac{1}{2}ax - \frac{1}{16}a^2x^2 \ldots\right), \]
and (b) can be applied.

(e) Hence, we have the conclusion seeming very strange to us at first sight, but less now: \( U, U \ast U, U \ast U \ast U, \ldots \) may have linear pieces near the origin.

4.4. \( G \)-distributions concentrated and very regular on some interval

Let \( G \) be very regular near 0. Then a unique very regular function \( G^0 \) exists such that \( G = G^0 \) near 0. Let
\[ b = \sup\{c : G = G^0 \text{ on } [0, c]\}. \]

Then \( G = G^0 \) if \( b = \infty \). This case may occur (see 4.9). Let us assume here that \( b \) is finite. Then \( G = G^0 \) on \([0, b]\), because \( G \) and \( G^0 \) are continuous. Every interval \([b, b + \varepsilon]\) contains a point \( x \) such that \( (G(x) \neq G^0(x)) \). We may call \( b \) the first singularity of \( G \). Beyond \( b \), the graphs of \( G \) and \( G^0 \) separate. On \([b, \infty[\), \( G \) remains a distribution, and \( G^0 \) remains very regular. Similarly, beyond \( b \) the graphs of \( U \) and \( U^0 \) separate. On \([b, \infty[\), \( U \) remains a distribution and \( U^0 \) remains very regular.

The approach by very regular functions seems useless beyond the first singularity of \( G \). If \( c \) is an atom of the claim-size distribution \( F \), the first singularity of \( G \) must be in \([0, c]\), because then \( G'(c - ) \neq G'(c + ) \) whereas \( G^0 \) has derivatives of all orders, hence continuous derivatives of all orders.

However, if \( G \) is concentrated and very regular on \([0, b]\) then the consideration of very regular functions may furnish \( U(x) \) for all \( x \in \mathbb{R} \), even if \( b \) is an atom of \( F \). The trick consist in the partition of \( \mathbb{R}_+ \) in intervals \([kb, kb + b]\) \((k = 0, 1, \ldots)\) and in the treatment, separately, of each interval, by very regular functions. It is worked out in the following Theorems 7 and 8.

**Theorem 9.** (General case). In classical risk theory, let us assume the following: \( G \) is concentrated on the compact interval \([0, b]\). \( G^0 \) is any \( BV \)-function coinciding with \( G \) on \([0, b]\). The function \( G^0_b \) is defined by
\[ G^0_b(x) = G^0(b + x) - G^0(b) \quad (x \in \mathbb{R}_+). \]
The functions \( V_k(k \geq 0) \) satisfy the renewal equations
\[ V_0 = p + qV_0 \ast G^0, \quad V_k = qV_k \ast G^0 - qV_{k-1} \ast G^0_b \quad (k \geq 1). \]

Then
\[ U(nb + x) = \sum_{k=0}^{n} V_k((n - k)b + x) \quad (n \geq 0, 0 \leq x \leq b). \]

**Proof.** We define \( U_k \) by
\[ U_k(x) = U(kb + x) \quad (k \geq 0, x \in \mathbb{R}_+). \]
Then

$$(U \ast G)(kb + x) = \int_{0}^{kb + x} U(kb + x - y) \, dG(y)$$

$$= \int_{0}^{x} U_k(y) \, dG(y) + \int_{x}^{kb + x} U(kb + x - y) \, dG(y)$$

$$= (U_k \ast G)(x) + \int_{x}^{b} U_{k-1}(b + x - y) \, dG(y) \quad (0 \leq x \leq b).$$

Taking the equation $U = p + qU \ast G$ at $kb + x$, we obtain

$$U_k(x) = p + q (U_k \ast G)(x) + q \int_{x}^{b} U_{k-1}(b + x - y) \, dG(y) \quad (k \geq 0, 0 \leq x \leq b).$$

(23)

For $k = 0$, it must be understood that there is no last term, or that $U_{-1} = 0$.

We define $U_k^0$ everywhere by

$$U_k^0(x) = \sum_{j=0}^{k} V_j((k-j)b + x) \quad (k \geq 0).$$

Then, everywhere

$$V_k(x) = U_k^0(x) - U_{k-1}^0(b + x) \quad (k \geq 0),$$

if we agree that $U_{-1}^0 = 0$.

For $k \geq 1$, we have by (21)

$$\frac{1}{q} (U_k^0(x) - U_{k-1}^0(b + x)) = \int_{0}^{x} U_k^0 - \int_{0}^{x} U_{k-1}^0 - \int_{b}^{b+x} U_{k-1}^0 + \int_{b}^{b+x} U_{k-2}^0$$

(24)

where we abbreviate

$$\int U_{k-1}^0 = \int U_{k-1}^0((kb + x) \, dG^0(y),$$

and use

$$\int_{0}^{x} f(y) \, dG^0(y) = \int_{b}^{b+x} f(y-b) \, dG^0(y).$$

Repeating (24) for the couples $(k-1, b + x)$, $(k-2, 2b + x), \ldots, (1, (k-1)b + x)$ and completing by (20) at the point $kb + x$, we have

$$\frac{1}{q} (U_{k-1}^0(b + x) - U_{k-2}^0(2b + x)) = \int_{0}^{b+x} U_{k-1}^0 - \int_{0}^{b+x} U_{k-2}^0 - \int_{b}^{2b+x} U_{k-2}^0 + \int_{b}^{2b+x} U_{k-3}^0$$

$$\frac{1}{q} (U_{k-2}(2b + x) - U_{k-3}(3b + x)) = \int_{0}^{2b+x} U_{k-2}^0 - \int_{0}^{2b+x} U_{k-3}^0 - \int_{b}^{3b+x} U_{k-3}^0 + \int_{b}^{3b+x} U_{k-4}^0$$

$$\frac{1}{q} (U_{k-3}(3b + x) - U_{k-4}(4b + x)) = \int_{0}^{3b+x} U_{k-3}^0 - \int_{0}^{3b+x} U_{k-4}^0 - \int_{b}^{4b+x} U_{k-4}^0 + \int_{b}^{4b+x} U_{k-5}^0$$

\ldots \ldots
\[
\frac{1}{q} \left( U^0_2((k-2)b + x) - U^0_1((k-1)b + x) \right)

= \int_0^{(k-2)b + x} U^0_2 - \int_0^{(k-2)b + x} U^0_1 - \int_b^{(k-1)b + x} U^0_1 + \int_b^{(k-1)b + x} U^0_0
\]

\[
\frac{1}{q} \left( U^0_1((k-1)b + x) - U^0_0(kb + x) \right) = \int_0^{(k-1)b + x} U^0_1 - \int_0^{(k-1)b + x} U^0_0 - \int_b^{kb + x} U^0_0
\]

\[
\frac{1}{q} U^0_0(kb + x) = p + \int_0^{kb + x} U^0_0.
\]

Summing up these relations, including (24), we obtain

\[
U^0_k(x) = p + q(U^0_k * G^0)(x) + q \int_x^{b} U^0_{k-1}(y-x) \, dG^0(y) \quad (k \geq 0).
\]

For \( k = 0 \), this relation is (20) at the point \( x \). Relations (25) hold everywhere, in particular on \([0, b]\). But \( G = G^0 \) on \([0, b]\), and \( U_0, U_1, U_2, \ldots \) are successively determined on \([0, b]\) by the renewal equations (23). Comparing (23) and (25), we conclude that \( U_k = U^0_k \) on \([0, b]\) (see Theorem 6).

**Theorem 10.** (Very regular case). In classical risk theory, let us assume the following: \( G \) is concentrated and very regular on the compact interval \([0, b]\). \( G^0 \) is the very regular function coinciding with \( G \) on \([0, b]\).

The function \( G^0 \) is defined by \( G^0(x) = G^0(b + x) - G^0(b)(x \geq 0) \).

Then the renewal equations

\[
V_0 = p + qV_0 * G^0,
\]

\[
V_k = qV_k * G^0 - qV_{k-1} * G^0 \quad (k \geq 1),
\]

have, successively, very regular solutions \( V_k(k \geq 0) \), and

\[
U(nb + x) = \sum_{k=0}^{n} V_k((n-k)b + x) \quad (n \geq 0, 0 \leq x \leq b).
\]

The factorial transform of \( V_k(k \geq 0) \) is, near \( 0 \),

\[
TV_k = p(-q)^k(TG^0_b)^k(1 - qTG^0)^{-k-1} \quad (k \geq 0).
\]

**Proof.** The theorem results from the preceding one. Here \( G^0 \) is very regular. Hence Eq. (26) has a very regular solution \( V_0 \) by Theorem 8. For \( k = 1 \), (27) then becomes the very regular renewal equation for \( V_1 \),

\[
V_1 = qV_1 * G^0 - qV_0 * G^0,
\]

where \( V_0 * G^0 \) is very regular by Theorem 4. Introducing \( V_1 \) in (27) for \( k = 2 \), gives a very regular equation for \( V_2 \), with very regular solution \( V_2 \). Taking the factorial transforms of (26) and (27), relations for \( TV_0, \ldots, TV_k \) are obtained. Eliminating successively \( TV_0, \ldots, TV_{k-1} \), the relation (29) for \( TV_k \) subsists.

**Theorem 11.** In classical risk theory, let the claim-size distribution \( F \) be concentrated on the compact interval \([0, b]\), and let it be very regular on \([0, b]\). Then \( G \) is very regular on \([0, b]\).
Proof. The function $f = 1 - F$ is very regular on $[0, b]$. Let

$$f^0(x) = \sum_k a_k x^k \quad (x \in \mathbb{R}_+)$$

be the very regular function coinciding with $f$ on $[0, b]$. Then

$$G(x) = \frac{1}{\mu} \int_0^x f(y) \, dy = \frac{1}{\mu} \sum_{k+1} \frac{1}{a_k} x^{k+1} \quad (0 \leq x \leq b),$$

the validity of the relation for $x = b$ resulting from the continuity of $G$ at $b$.

Let us define $G^0$ by

$$G^0(x) = \frac{1}{\mu} \sum_{k+1} \frac{1}{a_k} x^{k+1} \quad (x \in \mathbb{R}_+).$$

Then the very regularity of $G^0$ results from the very regularity of $f^0$, by an examination of the radius of convergence of $G^0$ and $\mathcal{T}G^0$. Hence $G$ is very regular on $[0, b]$, because $G$ and the very regular function $G^0$ coincide on $[0, b]$.

Remark. By the preceding theorem, the truncation of a claim-size distribution at $b$, and the introduction of an atom at $b$, cannot be the reason of non applicability of Theorem 10.

4.5. The use of other function extensions and other transforms

Let $F$ be concentrated on $[0, b]$, and let

$$F(x) = 1 - (1+x)^{-2} \quad (0 < x < b).$$

Then

$$G(x) = \frac{1}{\mu} (1 - (1+x)^{-1})$$

on $[0, b]$. We can define $G^0$ on $\mathbb{R}_+$ by the last member of (30) and apply Theorem 9 with this $G^0$. Here $G$ is not very regular on $[0, b]$ and Theorem 10 does not apply.

But instead of factorial transforms, we could use Laplace transforms, and enunciate the corresponding version of Theorem 10 (using some adapted version of Theorem 8).

Although Laplace transforms apply to a very wide class of functions, we did not pursue in this direction for two reasons.

The first is that, in connection with Schmitter's problem (see 4.3), we tried to prove that $U * U$ cannot have linear pieces, but we constructed an example of the contrary by exploiting the simple correspondence between $U$ and $\mathcal{T}U$ near 0. For Laplace transforms, this correspondence does not exist. (There the behavior of the transform near 0, $\infty$ is related to the behavior of the function near $\infty$, 0 respectively).

The second is that our aim was the obtention of explicit analytic results and this is rather easy and systematic with the factorial transforms.

By the way, we observe that Theorem 9 is correct for $G^0 = G$. In that case the solution $V_0$ of (20) is $U$, and $0 = V_1 = V_2 = \ldots$ are solution of (21) for $k = 1, 2, \ldots$ successively. The theorem is not contradicted for $G^0 = G$, but it learns nothing in that case.
4.6. The general result

We recall that an algebra of real (complex) functions of \( x \in \mathbb{R}_+ \) is a family of functions closed under the following operations:

(a) multiplication of a real (complex) scalar by a function;
(b) addition of two functions;
(c) multiplication of two functions.

The algebra generated by a given family \( \mathcal{F} \) of functions on \( \mathbb{R}_+ \) is the smallest algebra containing that family. It is the family of functions that can be constructed, starting from the functions of \( \mathcal{F} \), by a finite number of repeated applications of the steps (a), (b), (c), to functions obtained previously in the process.

**Theorem 12.** Let \( \mathcal{A} \) be the algebra of real functions of \( x \in \mathbb{R}_+ \) generated by the polynomials with real coefficients, the trigonometric functions \( \sin ax (a \in \mathbb{R}) \), \( \cos ax (a \in \mathbb{R}) \) and the exponential functions \( e^{ax} (a \in \mathbb{R}) \).

(a) Let \( F \) be a claim-size distribution on \( \mathbb{R}_+ \), belonging to \( \mathcal{A} \). Then \( U \) belongs to \( \mathcal{A} \), and \( U \) can be found explicitly as shown in the following proof.

(b) Let \( F \) be a claim-size distribution concentrated on the compact interval \( [0, b] \), coinciding with a function \( F^0 \in \mathcal{A} \) on \( [0, b] \). Then each function \( U(nb + x | n \geq 0) \) of \( x \), coincides with some function \( U^0_n \in \mathcal{A} \) on \( [0, b] \), and \( U^0_n \) can be obtained explicitly as shown in the following proof.

**Proof.** Let \( \mathcal{B} \) be the algebra of complex functions of \( x \in \mathbb{R}_+ \) generated by the polynomials with complex coefficients and the exponential functions \( e^{cx} (c \text{ complex}) \). Then each function of \( \mathcal{B} \) can be displayed as a sum with a finite number of terms

\[
\sum_k d_k x^{n_k} e^{c_k x} \quad (d_k \text{ complex}, c_k \text{ complex}, n_k \text{ integer}, n_k \geq 0). \tag{31}
\]

We have \( \mathcal{A} \subset \mathcal{B} \). The functions of \( \mathcal{B} \) are very regular and their factorial transforms are rational functions (see appendix). If \( f \in \mathcal{B} \), and \( f \) is real-valued, then \( f \in \mathcal{A} \).

(a) The claim-size distribution \( F \) can be displayed as (31). Then it is clear that \( G \in \mathcal{B} \). Being real-valued \( G \) is in \( \mathcal{A} \). Applying \( T \) to the basic renewal equation \( U = p + qU + G \), we obtain

\[
TU = \frac{p}{1 - qTG}.
\]

\( TG \) is a rational function. Hence \( TU \) also is a rational function, \( (TU)(0) \neq \infty \). Then \( TU \) can be inverted by the method of the appendix. The result is a function \( U \in \mathcal{B} \). Being real-valued, \( U \) belongs to \( \mathcal{A} \).

(b) Let \( G^0 \) be defined by

\[
G^0(x) = \frac{1}{\mu} \int_0^x (1 - F^0(y)) \, dy \quad (x \in \mathbb{R}_+).
\]

Then \( G^0 \) is the very regular extension of the restriction of \( G \) to \( [0, b] \) (see proof of Theorem 11). We have \( G^0 \in \mathcal{B}, G^0_b \in \mathcal{B} \). Hence \( TG^0 \) and \( TG^0_b \) are rational functions. By (29) \( TV_k \) is a rational function not infinite at 0. We can invert it by the method of the appendix. The result is a function \( V_k \in \mathcal{B} \). Being real-valued, \( V_k \) belongs to \( \mathcal{A} \). Then the last member of (28), say \( U^0_n(x) \), belongs to \( \mathcal{A} \).

**Remark.** The method of the preceding proof works in full generality. But it is only in exceptional cases that the obtention of the general \( V_k \) is easy.
If \( F \), in part (b) of the preceding theorem, results from the truncation at a large \( b \), of a distribution on \( \mathbb{R}_+ \), it may be sufficient to calculate only the first two or three functions of the sequence \( V_0, V_1, \ldots \), for practical purposes.

### 4.7. Illustrations with bounded claims

In the following examples, we illustrate part (b) of the preceding theorem. \( F \) is concentrated on \([0, b]\). Hence \( F = 1 \) on \([b, \infty[\). We have

\[
\mu = \int_0^b (1 - F(y)) \, dy,
\]

\[
G(x) = \frac{1}{\mu} \int_0^x (1 - F(y)) \, dy \quad (x \in \mathbb{R}_+).
\]

Hence \( G = 1 \) on \([b, \infty[\). It is not necessary to compute \( \mu \) before starting the calculation of \( G \). Indeed, \( G \) can be calculated with unspecified \( \mu \), and then \( \mu \) results from the equality \( G(b) = 1 \).

In all the examples, \( F \) is very regular on \([0, b]\) and \( G \) is very regular on \([0, b]\). We denote by \( G^0 \) the very regular function coinciding with \( G \) on \([0, b]\). \( G^0 \) has the same analytic expression on \( \mathbb{R}_+ \) as \( G \) on \([0, b]\). We obtain \( TV_k \) from (29) and then we invert by the method of the appendix. Then the probability of non-ruin results from (28).

The expression for a transform, say \( Tf(x) \), is always valid for \( x \) near 0. Then the expression for \( f(x) \) is valid everywhere.

**Example 1.** \( F \) concentrated at \( b \). Here \( F = 0 \) on \([0, b[\), \( G(x) = x / b \) \((0 \leq x \leq b)\), \( G^0(x) = x / b = G^0(x) \),

\[
TV_k(x) = p(-q)^k \frac{1}{b^k} x^k \left(1 - \frac{x}{b}\right)^{-k-1},
\]

\[
V_k(x) = pq^k \frac{1}{k!} (-1)^k \left(\frac{x}{b}\right)^k e^{x/b},
\]

\[
U(nb + x) = \sum_{k=0}^{n} pq^k (-1)^k \frac{1}{k!} \left(n - k + \frac{x}{b}\right)^k e^{q(n-k+x/b)} \quad (n \geq 0, 0 \leq x \leq b).
\]

This example is classical. See the article by Segerdahl in Grenander (1959).

**Example 2.** \( F \) uniform on \([0, b]\). Here we take \( F \) concentrated on \([0, b]\), \( F(x) = x / b \) \((0 \leq x \leq b)\). Then \( \mu = b/2 \),

\[
G(x) = 2 \frac{x}{b} - \left(\frac{x}{b}\right)^2 \quad (0 \leq x \leq b)
\]

and \( G^0(x) \) has the same expression on \( \mathbb{R}_+ \). Then

\[
G^0_b(x) = \left(\frac{x}{b}\right)^2,
\]

\[
TV_k(x) = p2^k q^k b^{-2k} x^{2k} \left(1 - 2q \frac{x}{b} + 2q \left(\frac{x}{b}\right)^2\right)^{-k-1}.
\]
The polynomial \( 1 - 2q(x/b) + 2q(x/b)^2 \) has two complex conjugate roots. It can be displayed as
\[
(1 - \alpha x)(1 - \beta x) = (1 - sx)^2 + t^2 x^2,
\]
where \( \alpha = s + ti, \quad \beta = s - ti, \quad s \in \mathbb{R}, \quad t \in \mathbb{R}. \)

Then
\[
V_k(x) = p 2^k q^k b^{-2k} W_k(s, t, x)
\]
where the entire function \( W_k(s, t, x) \) of \( x \), with real parameters \( s, t \) is defined by
\[
W_k(s, t, x) = T^{-1}\left( x^{2k}((1-sx)^2 + t^2 x^2)^{-k-1} \right).
\]

Then
\[
U(nb + x) = p 2^k q^k b^{-2k} W_k(s, t, (n - k)b + x) \quad (n \geq 0, 0 \leq x \leq b).
\]

Let us now indicate how \( W_k(s, t, x) \) can be obtained. We have
\[
\frac{1}{(1-sx)^2 + t^2 x^2} = \frac{1}{(1-\alpha x)(1-\beta x)} = \frac{1}{\alpha - \beta} \left( \frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x} \right).
\]

Hence
\[
W_0(s, t, x) = \frac{1}{\alpha - \beta} (\alpha e^{sx} - \beta e^{\beta x}).
\]

Eliminating the imaginary quantities, we obtain (see (A.14))
\[
W_0(s, t, x) = e^{sx} \left( \frac{\sin tx}{t} + \cos tx \right) \quad (x \in \mathbb{R}_+).
\]

From (33) results
\[
W_{k+1}(s, t, x) = -\frac{1}{2(k+1)t} \frac{\partial}{\partial t} W_k(s, t, x) \quad (k \geq 0).
\]

From (35) and (36) we successively obtain \( W_1, W_2, \ldots \) But the expressions are soon cumbersome. An easy general expression for \( W_k(s, t, x) \) can be obtained as follows. In the last member of (35), let us use the expansions
\[
\sin \frac{tx}{t} = \sum_{k \geq 0} (-1)^k \frac{t^{2k} x^{2k+1}}{(2k+1)!},
\]
\[
\cos \frac{tx}{t} = \sum_{k \geq 0} (-1)^k \frac{(tx)^{2k}}{(2k)!}.
\]

Then the application of (36) becomes easy, because the successive partial derivatives in \( t \) can be taken under the \( \sum \) symbol.

**Example 3.** \( F \) truncated exponential. Here we assume \( F \) concentrated on \([0, b]\) and
\[
F(x) = 1 - a e^{-cx} \quad (0 \leq x < b), \quad 0 < a \leq 1, \quad c > 0.
\]

Then
\[
G(x) = \frac{a}{\mu c} (1 - e^{-cx}) \quad (0 \leq x \leq b)
\]
and \( G^0(x) \) has the same expression on \( \mathbb{R}_+ \). Then
\[
G^0_b(x) = e^{-cx} G^0(x), \quad TG^0(x) = \frac{a}{\mu} \cdot \frac{x}{1+cx},
\]
\[
TV_k(x) = p(-q)^k e^{-kbc} \left( \frac{a}{\mu} \cdot \frac{x}{1+cx} \right)^k \left( 1 - q \frac{a}{\mu} \cdot \frac{x}{1+cx} \right)^{-k-1}
\]
\[
= p(-q)^k e^{-kbc} \frac{x^k}{(\mu + (\mu c - qa)x)^{k+1}}.
\]
We have to separate two cases.

**Case** \( \mu c = qa \). Then
\[
TV_k(x) = p(-q)^k e^{-kbc} a^k \mu^{-k} x^k (1+cx),
\]
\[
V_k(x) = p(-q)^k e^{-kbc} a^k \mu^{-k} \left( \frac{x^k}{k!} + c \frac{x^{k+1}}{(k+1)!} \right),
\]
\[
U(nb + x) = p \sum_{k=0}^{n} (-q)^k e^{-kbc} a^k \mu^{-k} \left( \frac{1}{k!} ((n-k)b + x)^k + \frac{c}{(k+1)!} ((n-k)b + x)^{k+1} \right)
\]
\[
( n \geq 0, 0 \leq x \leq b )
\]

**Case** \( \mu c \neq qa \). Let \( t = c - qa / \mu \). Then by (38) and (A.8) of the appendix,
\[
TV_k(x) = p(-q)^k e^{-kbc} a^k \mu^{-k} \left( \frac{x^k}{(1+ix)^{k+1}} + c \frac{x^{k+1}}{1+ix} \right),
\]
\[
V_k(x) = p(-q)^k e^{-kbc} a^k \mu^{-k} \left( \frac{x^k}{k!} e^{-ix} + c \frac{x^{k+1}}{(k+1)!} \sum_{j=0}^{k} \frac{i^j x^j}{j!} \right) ( k \geq 0, x \in \mathbb{R}_+ ).
\]

Substitution in (28) furnishes the explicit analytic expression for the probability of non-ruin.

In the article ‘A survey of results in the collective theory of risk’ in Grenander (1959), Segerdahl indicates the explicit analytic probability of ruin for a subfamily of claim-size distributions of the family defined by (37).

**Example 4.** Sinus claim-size density. Here we take \( F \) concentrated on \([0, \pi/2]\), \( b = \pi/2 \),
\[
F(x) = \int_0^x \sin y \, dy = 1 - \cos x \quad (0 \leq x \leq \pi/2).
\]

Then
\[
G^0(x) = \sin x, \quad G^0_b(x) = \cos x - 1,
\]
\[
T \cos x = \frac{1}{2} (T e^{ix} + T e^{-ix}) = \frac{1}{2} \left( \frac{1}{1-ix} + \frac{1}{1+ix} \right) = \frac{1}{1+x^2},
\]
\[
T \sin x = \frac{1}{2i} (T e^{ix} - T e^{-ix}) = \frac{1}{2i} \left( \frac{1}{1-ix} - \frac{1}{1+ix} \right) = \frac{x}{1+x^2},
\]
The polynomial $1 - qx + x^2$ has conjugate complex roots. As in Example 2, it can be displayed as

$$1 - qx + x^2 = (1 - sx)^2 + t^2 x^2, \quad s, t \in \mathbb{R}.$$ 

Then, using (A.7) twice,

$$V_k(x) = pq^k \left( W_k(s, t, x) + W_k^{II}(s, t, x) \right)$$ 

where the entire function $W_k(s, t, x)$ is defined in Example 2, and where

$$W_k^{II}(s, t, x) = \int_0^x W_k(s, t, y) \, dy, \quad W_k^{I}(s, t, y) = \int_0^y W_k(s, t, z) \, dz.$$ 

From (35), (36), (41) and (42), the function $V_k(x)$ of (41) can be found explicitly. Substitution in (28) gives the analytic explicit expression for $U(nb + x)$. But the expressions are soon very cumbersome. An easy general expression is found for $V_k(x)$ if $(1/t)$ sin $tx$ and cos $tx$ are replaced by their power series expansions in (35).

4.8. Unbounded claims

Let $\psi(x) = 1 - U(x)$ be the probability of ruin corresponding to the initial risk reserve $x$. Part (b) of the following theorem gives a more precise version of part (a) of the Theorem 12. The family $\mathcal{A}$ is the algebra defined there.

From the following theorem results that the adjustment coefficient equals $r = -1/x_0$ where $x_0$ is the smallest root of the equation $1 = qTG(x)$, if the claim-size distribution $F$ belongs to $\mathcal{A}$.

**Theorem 13.** Let the claim-size distribution $F$ be very regular on $\mathbb{R}_+$. 

(a) $\psi$ is very regular on $\mathbb{R}_+$ and

$$T\psi(x) = q \frac{1 - TG(x)}{1 - qTG(x)} \text{ near } 0,$$ 

$$TG(x) = \frac{1}{\mu} x(1 - TF(x)) \text{ near } 0.$$ 

(b) Let $F \in \mathcal{A}$. Then $\psi(x)$ is a sum of terms corresponding to the roots of the equation

$$1 = qTG(x)$$ 

in the following way. The terms corresponding to the real root $c$ of multiplicity $j$ are

$$a_0 e^{-rx}, a_1 x e^{-rx}, \ldots, a_{j-1} x^{j-1} e^{-rx},$$
where \( r = -1/c \) and \( a_0, \ldots, a_{j-1} \in \mathbb{R} \). The terms corresponding to the couple of conjugate roots \( \gamma, \delta \) of multiplicity \( k \) are

\[
\begin{align*}
&b_0 \cos(tx) e^{-sx}, b_1 x \cos(tx) e^{-sx}, \ldots, b_{k-1} x^{k-1} \cos(tx) e^{-sx}, \\
&c_0 \sin(tx) e^{-sx}, c_1 x \sin(tx) e^{-sx}, \ldots, c_{k-1} x^{k-1} \sin(tx) e^{-sx},
\end{align*}
\]

where

\[
-\frac{1}{\gamma} = s + it, -\frac{1}{\delta} = (s - it), \quad s, t \in \mathbb{R}
\]

and \( b_0, c_0, b_1, c_1, \ldots, b_{k-1}, c_{k-1} \in \mathbb{R} \).

One of the terms \( ae^{-rx} \) is Cramér's asymptotic expression for \( \psi(x) \) and the adjustment coefficient \( r \) equals \(-1/x_0\), where \( x_0 \) is the smallest real root of (45). This root is a simple root.

Eq. (45) has one real root at least, and all real roots are strictly negative. The real part of each complex root of (45) is strictly negative.

**Proof.** (a) \( U \) is a solution of the renewal equation \( U = p + q U \ast G \) and \( G \) is very regular. Hence, \( U \) is very regular by Theorem 1. Then \( \psi \) is very regular and it satisfies the renewal equation \( \psi = q(1 - G) + q\psi \ast G \). Taking factorial transforms, we obtain (43). Relation (44) results from (16) and (A.6).

(b) Now \( TG(x) \) is a rational function (see proof of Theorem 12). From (43) results that \( T\psi(x) \) is a rational function. Let \( T\psi(x) = P(x)/Q(x) \), where \( P(x) \) and \( Q(x) \) are polynomials without common factor. Then \( \psi(x) = T^{-1}(P(x)/Q(x)) \).

The degree of \( P \) is strictly less than the degree of \( Q \). Indeed, let us suppose the contrary. Then the division of \( P(x) \) by \( Q(x) \) furnishes terms \( ax^k (k \geq 0) \), with corresponding terms \( T^{-1}(ax^k) = ax^k/k! \) in \( \psi(x) \). But this contradicts that \( \psi(x) \rightarrow 0 \) for \( x \rightarrow \infty \).

Then everything results from Theorem A.2 of section A.5 of the appendix, taking the following remarks into account.

The equations \( Q(x) = 0 \) and \( 1 = qT G(x) \) are equivalent.

If

\[
Q(x) = a(x - x_1)^{k_1} \ldots (x - x_n)^{k_n} = b(1 + \gamma_1 x) \ldots (1 + \gamma_n x),
\]

then \( \gamma_j = -1/x_j \) (\( j = 1, \ldots, k \)), showing the relation between the root \( x_j \) and the factor \((1 + \gamma_j x_j)\).

In the factors \( e^{-rx}, e^{-sx} \) occurring in the terms of \( \psi(x) \), we must have \( r > 0, s > 0 \), because \( \psi(x) \rightarrow 0 \) for \( x \rightarrow \infty \).

The adjustment coefficient \( r \) exists, and Cramér's asymptotic expression is \( ae^{-rx} \) for some \( a \in \mathbb{R} \). The rest of the theorem follows from the meaning of this asymptotic expression.

### 4.9 Illustrations with unbounded claims

**Example 1.** \( F \) exponential-sinus. Here we take

\[
F(x) = 1 - e^{-x}(1 + 2a \sin x) \quad (x \in \mathbb{R}_+).
\]

This is a probability distribution if \( |a| \leq \frac{1}{4} \), with \( \mu = 1 + a \). Using

\[
T(e^{-x} \sin x) = \frac{x}{1 + 2x + 2x^2}
\]
and (44), we have
\[ TG(x) = x \frac{c + 2x + 2x^2}{(1 + x)(1 + 2x + 2x^2)}, \]
where \( c = 1/\mu \). Then by (43), \( T\psi(x) = qP(x)/Q(x) \), where
\[ P(x) = 1 + (3 - c)x + 2x^2, \quad Q(x) = 1 + (3 - cq)x + 2(2 - q)x^2 + 2(1 - q)x^3. \]
We suppose that \( Q(x) \) has one real root and two complex conjugate roots. This happened to be the case for all the numerical values that we tested, and it is obvious for \( q \) near 0.
Then \( Q(x) \) can be displayed as
\[ Q(x) = (1 + \alpha x)(1 + \beta x)(1 + \gamma x) \]
where
\[ \alpha = s + it, \quad \beta = s - it, \quad s > 0, \quad 0 < t \in \mathbb{R}, \quad r > 0. \]
By (A.18)
\[ \psi(x) = A e^{-rx} + B \cos(tx) e^{-sx} - C \sin(tx) e^{-sx} \quad (x \in \mathbb{R}_+) \]
where \( r, s, t, A, B, C \) are easily calculated.
The term \( Ae^{-rx} \) is Cramér’s asymptotic expression for \( \psi(x) \).

Example 2. F convolution of exponentials. Here we take \( F = F_1 * F_2 * F_3 \),
\[ F_j(x) = 1 - e^{-x/m_j} \quad (x \in \mathbb{R}_+; j = 1, 2, 3), \quad m_j > 0, \quad m_1 \neq m_2 \neq m_3 \neq m_1. \]
We set \( S_1 = m_1 + m_2 + m_3, \quad S_2 = m_1 m_2 + m_2 m_3 + m_3 m_1, \quad S_3 = m_1 m_2 m_3 \). The first moment of \( F_j \) is \( m_j \).
Hence \( \mu = m_1 + m_2 + m_3 = S_1 \). By (44),
\[ TG(x) = 1 - \frac{q}{\mu} x(1 - TF_1(x).TF_2(x).TF_3(x)), \]
where
\[ TF_j(x) = \frac{x}{x + m_j}. \]
Then, by (43), \( T\psi(x) = qP(x)/Q(x) \), where
\[ Q(x) = S_1 S_3 + (S_1 S_2 - qS_3)x + (S_1^2 - qS_2)x^2 + (1 - q)S_1 x^3, \]
\[ P(x) = S_1 S_3 + (S_1 S_2 - S_3)x + (S_1^2 - S_2)x^2. \]
We assume that \( Q(x) \) has three different real roots.
Then \( Q(x) \) can be displayed as
\[ Q(x) = S_1 S_3(1 + rx)(1 + sx)(1 + tx), \]
with \( 0 < r < s < t \).
By (A.15)
\[ \psi(x) = a e^{-rx} + b e^{-sx} + c e^{-tx} \quad (x \in \mathbb{R}_+), \]
where \( r, s, t, a, b, c \) are easily calculated.
The term \( ae^{-rx} \) is Cramér’s asymptotic expression for \( \psi(x) \).
4.10. A completely general formula

**Theorem 14.** In the classical risk model, let \( H_j (j \geq 0) \) be the distribution defined by

\[
H_j (x) = \frac{1}{j!} \left( \frac{qx}{\mu} \right)^j e^{qx/\mu} \quad (x \in \mathbb{R}_+).
\]

Then

\[
U = \sum_{j=0}^{\infty} (-1)^j H_j \ast F^* = \sum_{j=0}^{\infty} (-1)^j F^* \ast H_j
\]

and these series converge absolutely at each point \( x \in \mathbb{R}_+ \).

**Proof.** By (16) and (A.5),

\[
G = \frac{1}{\mu} (1 - F) \ast I,
\]

where \( I \) is the identity function on \( \mathbb{R}_+ \). Then

\[
I^* (x) = \frac{x^k}{k!}.
\]

Let us define the function \( U_c (c) \) on \( \mathbb{R}_+ \), first for \( c > 0 \), by

\[
U_c (x) = p \sum_{k \geq 0} \left( \frac{q}{\mu} \right)^k \left( k \right) c^j F^* \ast I^* k
\]

Then, by monotone convergence (everything being positive),

\[
U_c (x) = p \sum_{k \geq 0} \sum_{j \geq k} \left( \frac{q}{\mu} \right)^k \left( k \right) c^j F^* \ast I^* k
\]

\[
= p \sum_{j \geq 0} \sum_{k \geq 0} \left( \frac{q}{\mu} \right)^k \left( k \right) c^j F^* \ast I^* (k+j)
\]

\[
= \sum_{j \geq 0} c^j F^* \ast H_j = \sum_{j \geq 0} c^j H_j \ast F^*,
\]

where

\[
H_j (x) = p \sum_{k \geq 0} \left( \frac{q}{\mu} \right)^{k+j} \left( k+j \right) \frac{1}{(k+j)!} x^{k+j}
\]

\[
= \frac{p}{j!} \left( \frac{q^j x}{\mu} \right) \sum_{k \geq 0} \frac{1}{k!} \left( \frac{q^k x}{\mu} \right)^j = \frac{p}{j!} \left( \frac{q^j x}{\mu} \right)^j e^{qx/\mu}.
\]
We notice that
\[ U(c)(x) = \sum_{j \geq 0} c^j (F^* * H_j)(x) \leq \sum_{j \geq 0} c^j H_j(x) = p \ e^{\phi x / \mu} \ e^{\omega x / \mu} < \infty. \]

Now we consider \( U(c) \) for \( c = -1 \). Then the developments leading to (49) are permitted by dominated convergence, because \( U(1)(x) < \infty \).

By (48) and Theorem 7,
\[ U = p \sum q^k G^* k = p \sum \left( \frac{q}{\mu} \right)^k (1 - F)^* k * I^* k = U(-1) = \sum_{j \geq 0} (-1)^j H_j * F^* j = \sum_{j \geq 0} (-1)^j F^* j * H_j. \]

**Corollary** (Shiu, 1988). Let \( F \) be concentrated on \( \{1, 2, \ldots \} \). Then
\[ U(x) = \sum_{k = 0}^{[x]} \sum_{j = 0}^{k} (-1)^j H_j(x - k)(F^* j(k) - F^* j(k - 1)). \]

where \([x]\) is the integer part of \( x \in \mathbb{R}_+ \).

**Proof.** The distributions \( F^* j \) are concentrated on \( \{0, 1, \ldots \} \). Hence
\[
U(x) = \sum_{j = 0}^{\infty} (-1)^j \int_{0}^{x} H_j(x - y) \ dF^* j(y) \\
= \sum_{j = 0}^{[x]} \sum_{k = 0}^{x} (-1)^j H_j(x - k)(F^* j(k) - F^* j(k - 1)) \\
= \sum_{k = 0}^{[x]} \sum_{j = 0}^{k} (-1)^j H_j(x - k)(F^* j(k) - F^* j(k - 1))
\]

where \( F^* j(k) = 0 \) for \( j > k \).

**Appendix. The calculation of inverse factorial transforms**

**A.1. Complex functions of a real variable**

Here we consider functions of \( x \in \mathbb{R} \), having everywhere, or only near 0, expansions \( \sum a_k x^k \) with complex coefficients \( a_k \). Several previous definitions (very regular functions, \( T, T^{-1}, \ldots \)) and results (1.2, 1.3, \ldots) extend obviously to these complex functions of the real variable \( x \). They serve as intermediates. The final results, in risk theory, are always real.

**A.2. Particular transforms**

We shall see how to invert rational functions. The method is based on the transforms, valid near 0,
\[
T \ e^{sx} = \frac{1}{1 - sx}, \quad T \left( \frac{1}{k!} x^k e^{sx} \right) = \frac{x^k}{(1 - sx)^{k+1}} \quad (k \geq 0) \quad (A.1)
\]
where \( s \) is a complex parameter. The proof is direct by the Maclaurin expansions for the exponential and for negative integer powers of the binomial \( 1 - sx \). The last formula of (A1) also results from the first by successive derivations in \( s \).
A.3. Inversion of a complex rational function

The problem of inversion of a rational function without pole at the origin is solved by (A.1) and the following theorem.

**Theorem A.1.** Let \( Q(x) \) be the polynomial

\[
Q(x) = (1 - s_1 x)^{k_1} (1 - s_2 x)^{k_2} \cdots (1 - s_n x)^{k_n}
\]

where \( s_j \neq s_k \) if \( j \neq k, s_j \neq 0 \)

and let \( P(x) \) be a polynomial with degree strictly less than the degree of \( Q(x) \). Then the rational function \( f(x) = P(x)/Q(x) \) can be displayed as a linear combination of the terms

\[
\frac{x^k}{(1-sx)^{k+1}} \quad (s = s_j, k < k_j, j = 1, 2, \ldots, n).
\]

(A.2)

**Proof.** From the theory of integration of rational functions, we know that

\[
f(x) = \frac{1}{(1-sx)^{m+1}} \quad (s = s_j, m < k_j, j = 1, 2, \ldots, n).
\]

(A.3)

Each term (A.3) is a linear combination of terms (A.2). Indeed,

\[
\frac{1}{(1-sx)^{m+1}} = \frac{1}{1-sx} \left( 1 + \frac{sx}{1-sx} \right)^m = \frac{1}{1-sx} \sum_{k=0}^{m} \binom{m}{k} \left( \frac{sx}{1-sx} \right)^k = \sum_{k=0}^{m} \left( \frac{m}{k} \right) \frac{s^k x^k}{(1-sx)^{k+1}}.
\]

A.4. Particular formulae

(a) If \( P(x) \) is a polynomial and \( s \neq 0 \) a complex number, then

\[
P(x) e^{-sx} \, dx = d \left( -\frac{1}{s} e^{-sx} \left( P(x) + \frac{1}{s} P'(x) + \frac{1}{s^2} P''(x) + \ldots \right) \right),
\]

by direct verification. The number of terms in the last member is finite.

(b) Let \( I \) be the identity function on \( \mathbb{R} \): \( I(x) = x \). Then

\[
(f * I)(x) = \int_0^x f(y) \, dy.
\]

(A.5)

By Theorem 4, this implies

\[
T\int_0^x f(y) \, dy = x.Tf(x) \quad (f \text{ very regular}).
\]

(A.6)

(c) If \( g \) has a power series expansion converging near 0, then

\[
T^{-1}(xg(x)) = \int_0^x T^{-1}g(y) \, dy.
\]

(A.7)

Indeed, applying \( T \) to the last member, we obtain, by (A.6) \( xTT^{-1}g(x) = xg(x) \).

(d) By (A.7), (A.1) and (A.4), we have

\[
T^{-1} \left( \frac{x}{1+tx} \right)^{k+1} = T^{-1} \left( x \cdot \frac{x^k}{(1+tx)^{k+1}} \right) = \int_0^x \frac{1}{k!} y^k e^{-ty} \, dy
\]

\[
= \frac{1}{t^{k+1}} \left( 1 - e^{-tx} \sum_{j=0}^{k} \frac{1}{j!} t^j x^j \right).
\]

(A.8)
A.5. Inversion of real rational functions

By Theorem A.1, \( f(x) \) can be displayed as

\[
\frac{P(x)}{Q(x)} = \sum_{jk} a_{jk} \frac{x^k}{(1-s_j x)^{k+1}}.
\]

Let us multiply this relation by \( Q(x) \). Then we obtain a relation among polynomials. An identification of coefficients furnishes a linear system for the determination of the \( a_{jk} \). The consideration, in the relation among polynomials, of the particular values \( x = 0, x = 1/s_j (j = 1, \ldots, n) \) is helpful.

Let us assume now that \( f \) is real.

Let \((1-rx)^i\) be a factor of \( Q(x) \), \( r \in \mathbb{R} \). In the decomposition of \( f(x) \), the part corresponding to this factor is

\[
a_0 \frac{1}{1-rx} + a_1 \frac{x}{(1-rx)^2} + \ldots + a_{j-1} \frac{x^{j-1}}{(1-rx)^j}
\]

and its factorial inverse equals

\[
\left( a_0 + a_1 x + \ldots + a_{j-1} \frac{x^{j-1}}{(j-1)!} \right) e^{rx}.
\] (A.9)

Now, let \((1-ax)^k\) be a factor of \( Q(x) \), \( a = s + it, s \in \mathbb{R}, 0 \neq t \in \mathbb{R} \). Then \( Q(x) \) must also have the conjugate factor \((1-\bar{a}x)^k\), \( \bar{a} = s - it \). The part in the decomposition of \( f(x) \), corresponding to these factors is

\[
\left( b_0 \frac{x^{k-1}}{(1-ax)^k} \right) + \left( c_0 \frac{1}{1-\beta x} + \ldots + c_{k-1} \frac{x^{k-1}}{(1-\beta x)^k} \right)
\]

and its factorial inverse equals

\[
\left( b_0 + b_1 x + \ldots + b_{k-1} \frac{x^{k-1}}{(k-1)!} \right) e^{ax} + \left( c_0 + c_1 x + \ldots + c_{k-1} \frac{x^{k-1}}{(k-1)!} \right) e^{\beta x}
\]

\[
= \left( A(x) + iB(x) \right) e^{sx} \left( \cos(tx) + i \sin(tx) \right) + \left( C(x) + iD(x) \right) e^{sx} \left( \cos(tx) - i \sin(tx) \right)
\]

\[
= \left( A \cos tx - B \sin tx + C \cos tx + D \sin tx \right) e^{sx}
\]

\[
+ i \left( A \sin tx + B \cos tx - C \sin tx + D \cos tx \right) e^{sx},
\]

where \( A(x), B(x), C(x), D(x) \) are polynomials with degree strictly less than \( k \).

The imaginary parts must cancel, because \( T^{-1}f(x) \) is real, and there can be no cancellations with terms corresponding to other factors of \( Q(x) \).

For the same reason, the coefficients \( a_0, \ldots, a_{j-1} \) in (A.9) must be real.

Summing up, we have proved the following theorem.

Theorem A.2 To the factor \((1-rx)^i\) of \( Q(x) \) in Theorem A.1 corresponds in \( T^{-1}f(x) \) the term

\[
P(x) e^{rx},
\] (A.10)

where \( P(x) \) is a real polynomial of degree strictly less than \( j \). To the factors \((1-ax)^k, (1-\beta x)^k \), where \( \alpha = s + it, \beta = s - it, s \in \mathbb{R}, 0 \neq t \in \mathbb{R} \), corresponds the term

\[
(R(x) \cos(tx) + S(x) \sin(tx)) e^{sx},
\] (A.11)
where \( R(x), S(x) \) are real polynomials of degree strictly less than \( k \).

### A.6. Particular inverses of real rational functions

Hereafter we display the factorial inverse of particular real rational functions. They can be obtained by the general method explained at the beginning of A.5, and the subsequent elimination of the imaginary quantities. Another way to find them can be based on the last theorem, used with indeterminate coefficients. The latter result from an identification of the coefficients of 1, \( x, x^2 \).

These transforms are useful for the determination of ruin probabilities. In that case, if we write a denominator in the way \((1-sx)(1-tx)\), it appears that \( s \) and \( t \) are mostly negative. It is more convenient to display such a denominator as \((1+sx)(1+tx)\), for the applications.

We assume that all the displayed factors of the denominators are different. Hence, if \((1+sx)(1+tx)\) is written, then \( s \neq t \).

\[
\begin{align*}
T^{-1} \frac{a + bx}{(1 + sx)(1 + tx)} &= \frac{as - b}{s - t} e^{-sx} - \frac{at - b}{s - t} e^{-tx}, \\
T^{-1} \frac{a + bx}{(1 + sx)^2} &= a e^{-sx} - (as - b)x e^{-sx}, \\
T^{-1} \frac{a + bx}{(1 + \alpha x)(1 + \beta x)} &= \left( a \cos(tx) - (as - b) \frac{1}{t} \sin(tx) \right) e^{-sx}, \quad \alpha = s + it, \quad \beta = s - it, \\
T^{-1} \frac{a + bx + cx^2}{(1 + rx)(1 + sx)(1 + tx)} &= a \frac{r^2 - br + c}{(r - s)(r - t)} e^{-rx} + \frac{as^2 - bs + c}{(s - r)(s - t)} e^{-sx} + a \frac{t^2 - bt + c}{(t - r)(t - s)} e^{-tx}, \\
T^{-1} \frac{a + bx + cx^2}{(1 + sx)^2(1 + tx)} &= \frac{1}{(s - t)} \left( as^2 - 2a^2 + b + c \right) e^{-sx} - \frac{1}{s - t} (as^2 - bs + c)x e^{-sx} \\
&\quad + \frac{1}{(s - t)} \left( a^2 - bt + c \right) e^{-tx}, \\
T^{-1} \frac{a + bx + cx^2}{(1 + sx)^3} &= a e^{-sx} - (2as - b)x e^{-sx} + \frac{1}{2} (as^2 - bs + c)x^2 e^{-sx}, \\
T^{-1} \frac{a + bx + cx^2}{(1 + rx)(1 + \alpha x)(1 + \beta x)} &= A e^{-rx} + \left( B \cos(tx) - C \frac{1}{t} \sin(tx) \right) e^{-sx}, \\
\alpha &= s + it, \quad \beta = s - it,
\end{align*}
\]

where

\[
A = \frac{ar^2 - br + c}{(r - s)^2 + t^2}, \quad B = a - A, \quad C = a(s + r) + A(s - r) - b.
\]
References


