Upper stop-loss bounds for sums of possibly dependent risks with given means and variances

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Abstract

Consider non-negative random variables $X_1, \ldots, X_n$ whose marginal means and variances are known. The purpose of this paper is to compare two different strategies for finding an upper bound on the stop-loss premium $\pi(X_1 + \cdots + X_n, d) = E\{\max(0, X_1 + \cdots + X_n - d)\}$ that are valid for all retention amounts $d \geq 0$ in the absence of information concerning the type or degree of dependence between the risks $X_i$. One approach consists of maximizing the premium over all possible values $\rho_{ij} = \text{corr}(X_i, X_j)$, $1 \leq i < j \leq n$. As it turns out, however, a better solution exploits results of Dhaene et al. (Schweiz. Aktuarver. Mitt. (2000) 99) on the maximality of comonotonic risks in the stop-loss order. Explicit calculations and numerical illustrations of the proposed bounds are given. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

In actuarial mathematics, a risk is typically modeled by a random variable $X \geq 0$ representing the monetary value, either of (i) a single insurance claim, (ii) the sum of all claims for an individual contract, or (iii) a portfolio of contracts over a given time period. Of special interest is the degree of dangerousness of such a risk, once the amount $d \geq 0$ retained by the insured (or the first insurer) has been factored out. This can be measured, in particular, by the stop-loss premium

$$\pi(X, d) = E\{\max(0, X - d)\},$$

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for which it is useful to have upper bounds when the distribution of $X$ is not completely determined.

Beginning with the work of Bühlmann et al. (1977), there has been a great deal of literature on this subject, comprehensively reviewed by Hürlimann (1999). For example, when the only available information is

$$E(X) = \mu > 0 \quad \text{and} \quad \text{var}(X) = \sigma^2,$$

it is well known (cf., e.g., Theorem 2 of Jansen et al., 1986) that for all retention amounts $d \geq 0$, the best upper bound on $\pi(X,d)$ is given by

$$\pi^+_{\mu,\sigma^2}(d) = \begin{cases} \frac{\mu - d}{\mu^2 + \sigma^2}, & \text{if } 0 \leq d \leq \frac{\mu^2 + \sigma^2}{2\mu}, \\ \frac{\mu - d}{2} + \frac{1}{2} \sqrt{(d - \mu)^2 + \sigma^2}, & \text{if } d > \frac{\mu^2 + \sigma^2}{2\mu}. \end{cases} \tag{1}$$

For bounds under more restrictive hypotheses on the claim size distribution (e.g., additional moment conditions, finite range, unimodality, etc.), see De Vylder and Goovaerts (1982), Jansen et al. (1986) and Hürlimann (1996), among others.

The purpose of this paper is to examine strategies for finding upper bounds on $\pi(X,d)$ when $X = X_1 + \cdots + X_n$ in the presence of partial information on the marginal distributions of the $X_i$’s but no specific knowledge about their dependence structure. This problem arises when $X$ represents an aggregate claim whose components might exhibit stochastic dependence.

When the marginals are fully specified, Dhaene and Goovaerts (1996), Müller (1997), Bäuerle and Müller (1998) and Wang and Dhaene (1998), have shown that the best upper stop-loss bound is obtained when the $X_i$’s are comonotonic, i.e., when their underlying copula is the Fréchet upper bound. Hürlimann (1998) worked on the same problem, assuming that nothing of the marginals is known but the means, variances and correlation between two risks $X_1$ and $X_2$ that are diatomic, i.e., whose probability mass function is concentrated on two points.

Two different approaches for bounding the stop-loss premium $\pi(X,d)$ are considered in Section 2 when the first two moments of the $X_i$’s are given. A natural solution consists of maximizing the premium over all possible correlation structures between these risks. Another way to proceed is to exploit the fact, mentioned by Dhaene et al. (2000), that the sum of comonotonic risks $Y_1,\ldots,Y_n$ constitutes an upper bound on the sum of the $X_i$’s in the stop-loss order whenever $\pi(X_i,d) \leq \pi(Y_i,d)$ for all $d \geq 0$ and $1 \leq i \leq n$. Explicit calculations of the resulting upper bounds are described in Section 3 for the special case $n = 2$, and the better of the two solutions is identified in Section 4. A couple of illustrations are given in Section 5, and concluding remarks are made in Section 6.

2. Upper stop-loss bounds for an arbitrary sum of risks

Suppose that $X_1,\ldots,X_n$ are non-negative random variables with $E(X_i) = \mu_i > 0$ and $\text{var}(X_i) = \sigma_i^2 > 0$, $1 \leq i \leq n$. In this section, two natural approaches are described for deriving stop-loss bounds
on \(X = X_1 + \cdots + X_n\) in the absence of complete information concerning the joint distribution of the \(X_i\)’s.

**First approach.** A simple solution consists of fixing temporarily the values of \(\rho_{ij} = \text{corr}(X_i, X_j)\), \(1 \leq i < j \leq n\), so that

\[
E(X) = \mu = \sum_{i=1}^{n} \mu_i \quad \text{and} \quad \text{var}(X) = \sigma^2(\rho) = \sum_{i=1}^{n} \sigma_i^2 + 2 \sum_{i<j} \rho_{ij} \sigma_i \sigma_j.
\]

In view of (1), one can then assert that

\[
\pi(X, d) \leq \pi_{\mu, \sigma^2(\rho)}^+ (d),
\]

for all \(d \geq 0\), where the subscript makes explicit the dependence of the bound on the assumed vector \(\rho\) of values of the \(\rho_{ij}\)’s. If \(\mathcal{P}\) represents the set of all possible such vectors, an upper bound on \(\pi(X, d)\) that does not depend on the arbitrary choice of \(\rho\) is then given by

\[
\sup_{\rho \in \mathcal{P}} \pi_{\mu, \sigma^2(\rho)}^+ (d). \tag{2}
\]

**Second approach.** An alternative upper bound on \(\pi(X, d)\) inspired by the work of Dhaene et al. (2000) is given by

\[
\pi\{F_{1}^{-1}(U) + \cdots + F_{n}^{-1}(U), d\}, \tag{3}
\]

where \(U\) is a uniform random variable on the interval \((0, 1)\) and \(F_i = F_{\mu_i, \sigma_i^2}\) with

\[
F_{\mu, \sigma^2}(x) = \begin{cases} 
\frac{\sigma^2}{\mu^2 + \sigma^2}, & \text{if } 0 \leq x \leq \frac{\mu^2 + \sigma^2}{2\mu}, \\
\frac{1}{2} + \frac{1}{2} \frac{x - \mu}{\sqrt{(x - \mu)^2 + \sigma^2}}, & \text{if } x > \frac{\mu^2 + \sigma^2}{2\mu}.
\end{cases}
\]

Indeed, it is easy to check (e.g., using Theorem 1.2 on p. 22 of the book by Kaas et al., 1994) that \(\pi\{F_1^{-1}(U), d\} = \pi_{\mu_1, \sigma_1^2}^+ (d)\) yields the best upper bound on \(\pi(X_1, d)\), so that \(Y_i = F_1^{-1}(U)\) dominates \(X_i\) in the stop-loss order. Inequality (3) then follows from Theorem 5 of Dhaene et al. (2000), since the \(Y_i\)’s are comonotonic.

It is natural to ask which of quantities (2) or (3) yields the better bound. This issue is investigated next in the special case where \(X\) is the sum of two risks.

### 3. Computation of the bounds in the case \(n = 2\)

It is shown in the appendix that for arbitrary but fixed \(\mu_i\) and \(\sigma_i^2\), \(i = 1, 2\), one has

\[
\sup_{-1 \leq \rho \leq 1} \pi_{\mu_i, \sigma_i^2(\rho)}^+ (d) = \pi_{\mu, \sigma^2(1)}^+ (d),
\]

for all \(d \geq 0\). Accordingly, bound (2) on the stop-loss premium on a risk \(X = X_1 + X_2\) reduces to

\[
\pi_{\mu, \sigma^2(1)}^+ \quad \text{with} \quad \mu = \mu_1 + \mu_2 \quad \text{and} \quad \sigma = \sigma_1 + \sigma_2. \tag{4}
\]
To compute bound (3), assume without loss of generality that
\[ a_1 \equiv \frac{\sigma_1^2}{\mu_1^2 + \sigma_1^2} < \frac{\sigma_2^2}{\mu_2^2 + \sigma_2^2} \equiv a_2, \] (5)
and use the fact that
\[ F_{\mu,\sigma_1}^{-1}(u) = \begin{cases} 
0, & \text{if } 0 \leq u \leq \frac{\sigma^2}{\mu^2 + \sigma^2}, \\
\mu + \sigma \frac{2u - 1}{\sqrt{1 - (2u - 1)^2}}, & \text{if } u > \frac{\sigma^2}{\mu^2 + \sigma^2},
\end{cases} \]
to derive the cumulative distribution function of
\[ V = F_{\mu,\sigma_1}^{-1}(U) + F_{\mu,\sigma_2}^{-1}(U). \]
Writing \( F_V(v) = P(V \leq v) \) as
\[ P(V \leq v, U \leq a_1) + P(V \leq v, a_1 < U \leq a_2) + P(V \leq v, U > a_2), \]
one can see right away that the first term equals \( a_1 \), since \( U \leq a_1 \) implies \( F_{\mu,\sigma_1}^{-1}(U) = F_{\mu,\sigma_1}^{-1}(U) \equiv 0 \), and \( U \) is uniformly distributed on \((0,1)\). Similarly, the second term is simply
\[ P\{U \leq F_1(v), a_1 < U \leq a_2\} = P[a_1 < U \leq \min\{F_1(v), a_2\}] = \max\{0, \min\{F_1(v), a_2\} - a_1\}. \]
Furthermore, noting that for all \( a_2 < u \leq 1 \), one has
\[ F_{\mu,\sigma_1}^{-1}(u) + F_{\mu,\sigma_2}^{-1}(u) = F_{\mu,\sigma_1}^{-1}(u) \]
with \( \mu \) and \( \sigma \) given by (4), the third term can be reexpressed as
\[ P\{a_2 < U \leq F_{\mu,\sigma_2}(v)\} = \max\{0, F_{\mu,\sigma_2}(v) - a_2\}. \]
Letting
\[ b_1 = \lim_{v \downarrow a_1} F_1^{-1}(v) = \frac{\mu_1^2 + \sigma_1^2}{2\mu_1}, \]
\[ b_2 = F_1^{-1}(a_2) = \mu_1 + \frac{\sigma_1^2 - \mu_2^2}{\sigma_2} \]
and
\[ b_3 = F_{\mu,\sigma_2}^{-1}(a_2) = \mu + \frac{\sigma \sigma_2^2 - \mu_2^2}{2\mu_2}, \]
it follows from (5) that \( b_1 \leq b_2 \leq b_3 \), and one may then verify easily that
\[ F_V(v) = \begin{cases} 
a_1, & \text{if } 0 \leq v \leq b_1, \\
F_1(v), & \text{if } b_1 < v \leq b_2, \\
a_2, & \text{if } b_2 < v \leq b_3, \\
F_{\mu,\sigma_2}(v), & \text{if } b_3 < v < \infty. \end{cases} \]
Elementary calculations further yield
\[
\pi(V, d) = \begin{cases} 
\pi_{\mu_1, \sigma_1}^+(d) + \mu_2, & \text{if } 0 \leq d \leq b_2, \\
(b_3 - d)(1 - a_2) + \pi_{\mu_2, \sigma_2}^+(b_2), & \text{if } b_2 \leq d \leq b_3, \\
\pi_{\mu_2, \sigma_2}^+(d), & \text{if } d \geq b_3,
\end{cases}
\]
where \(\mu\) and \(\sigma\) are as defined in (4). That this bound is continuous in \(d\) follows in part from the fact that the middle term may be written alternatively as
\[
(b_3 - d)(1 - a_2) + \pi_{\mu_1, \sigma_1}^+(b_2) + \mu_2/2
\]
and that \((b_3 - b_2)(1 - a_2) = \mu_2/2\).

4. Comparison of the bounds when \(n = 2\)

In the limiting case \(a_1 = a_2\), it is easy to see that \(\sigma_1/\mu_1 = \sigma_2/\mu_2 = \sigma/\mu\), whence \(b_1 = b_2\) and
\[
a_1 = a_2 = a = \frac{\sigma^2}{\mu^2 + \sigma^2}.
\]
Since \(b_3\) is then equal to \((\mu^2 + \sigma^2)/(2\mu)\), it follows that \(F_V \equiv F_{\mu, \sigma^2}\), which implies that bounds (2) and (3) coincide in this case.

When \(a_1 < a_2\), however, bound (3) turns out to be different, and uniformly better, than bound (2). To prove this, first observe that the distribution functions \(F_{\mu, \sigma^2}\) and \(F_V\) associated with these bounds have the same expected value, namely \(\pi_{\mu, \sigma^2}(0) = \pi(V, 0) = \mu\). It is also clear that
\[
a_1 = F_V(v) \leq F_{\mu, \sigma^2}(v) = a, \quad 0 \leq v \leq b_1
\]
and
\[
a_2 = F_V(v) \geq F_{\mu, \sigma^2}(v), \quad b_2 \leq v \leq b_3,
\]
since the two curves are monotone increasing and coincide for all \(v \geq b_3\). If it can be shown that \(F_{\mu, \sigma^2}\) and \(F_V\) cross once and only once in the interval \([b_1, b_3]\), the inequality
\[
\pi(V, d) \leq \pi_{\mu, \sigma^2}^+(d), \quad d \geq 0,
\]
will then follow as a special case of Theorem 3.a.12 in Shaked and Shantikumar (1994). To this end, take \(a_1 \leq a_2\) as before, and introduce
\[
b = F_{\mu, \sigma^2}^{-1}(a) = \frac{\mu^2 + \sigma^2}{2\mu},
\]
whose value is comprised in the interval \((b_1, b_3]\). If \(b \geq b_2\), then \(f(v) = F_V(v) - F_{\mu, \sigma^2}(v)\) is continuous and strictly increasing on \([b_1, b_2]\) with \(f(b_1) = a_1 - a < 0\) and \(f(b_2) = a_2 - F_{\mu, \sigma^2}(b_2) > 0\), so that there can be only one root. If \(b < b_2\), two cases must be considered, according as \(F_V(b)\) is strictly smaller or larger than \(a\).
When $F_V(b) < a$, $f(v) = 0$ if and only if
\[
\varphi \left( \frac{v - \mu_1}{\sigma_1} \right) = \varphi \left( \frac{v - \mu}{\sigma} \right),
\]
where $\varphi(u) = u/\sqrt{u^2 + 1}$ is strictly monotone and increasing. There is, therefore, one and only one root.

When $F_V(b) > a$, there is a single root of $f$ on $[b_1, b]$, since $F_{\mu, \sigma}(v)$ is constant on that interval and hence $f$ is strictly increasing on this domain. Furthermore, any additional root of $f$ in the interval $[b, b_2]$ must satisfy relation (6), and is therefore unique, provided it exists. If one could find such a root, however, then one would still have $f(v) \geq 0$ for all $b \leq v \leq b_2$, or else $f(b_2)$ would be strictly negative, which is impossible. In this case again, therefore, $F_{\mu, \sigma}$ and $F_V$ cross once and only once, which completes the argument.

5. Examples

As an illustration of the above result, Fig. 1 displays the upper bound (3) as a function of the retention amount $d \geq 0$ when $X_1$ and $X_2$ have mean $\mu_1 = 175$, $\mu_2 = 150$ and variance $\sigma_1^2 = 61,250$ and $\sigma_2^2 = 45,000$, respectively. Since in this case $a_1 = a_2 = 2/3$, $\pi(V, d) = \pi_{\mu, \sigma}(d)$ with $\mu = \mu_1 + \mu_2 = 325$ and $\sigma^2 = (\sigma_1 + \sigma_2)^2 = 211,250$. This dashed line may be compared to the solid line corresponding to the exact value of $\pi(X, d)$ when $X_1$ and $X_2$ are comonotonic Pareto random variables with distribution function
\[
P(X_i \leq x) = 1 - \left( \frac{\lambda_i}{\lambda_i + x} \right)^x, \quad x = 4, \quad \lambda_1 = 525, \quad \lambda_2 = 450.
\]
As indicated by Wang and Dhaene (1998), among others, the latter is actually an upper bound on the stop-loss premium when the marginal distributions of the two risks are known to have Pareto distributions with these specific parameters but their joint dependence is uncertain.

![Fig. 1](image-url)
As a second example, the bounds (2) and (3) are compared in Fig. 2 with the exact value of \( \pi(X,d) \) when \( X_1 \) and \( X_2 \) are comonotonic and are distributed as

\[
F(x) = 1 - \exp\{ -2\sqrt{\gamma}(\sqrt{x + \beta} - \sqrt{\beta}) \}, \quad \beta, \gamma, x > 0,
\]

i.e., follow an exponential-inverse Gaussian distribution, whose mean and variance are given by

\[
\frac{1}{2\gamma} + \sqrt{\frac{\beta}{\gamma}} \quad \text{and} \quad \frac{5/4 + \beta\gamma + 2\sqrt{\beta\gamma}}{\gamma^2},
\]

respectively. (Note the typographical error in the formula for the variance given by Hesselager et al., 1998).
In Fig. 2, the solid line depicts $\pi(X, d)$ when $\mu_1 = 175$, $\mu_2 = 150$, $\sigma_1^2 = 40,625$ and $\sigma_2^2 = 42,500$. In this case, $a_1 \approx 0.5702$ and $a_2 \approx 0.6538$, so that $\pi(V, d)$ (dotted line) is uniformly smaller than $\pi_{\mu, \sigma^2}(d)$ (dashed line). Here, $\mu = \mu_1 + \mu_2 = 325$ and $\sigma^2 = (\sigma_1 + \sigma_2)^2 = 166,228.9$.

6. Discussion

In the two illustrations provided above, the bound (3) based only on the knowledge of the first two moments of the $X_i$’s is obviously more conservative than that corresponding to the case where the marginals are totally specified. With additional information concerning higher marginal moments of $X_1$ and $X_2$, however, it should be possible to improve the upper stop-loss bound derived in Section 4 by exploiting, say, results of Jansen et al. (1986). Following these authors, one could also use either Approach 1 or 2 to derive an upper stop-loss bound for the sum of two risks with finite support.

In the case $n > 2$, an advantage of bound (3) over bound (2) is that it can be more readily computed in explicit form. One may conjecture that it is also tighter, but considering the difficulty of maximizing $\pi_{\mu, \sigma^2}(d)$ over a multidimensional set $\mathcal{P}$ of possible correlation vectors $\rho$, a proof along the same lines as Section 4 does not seem practical. This issue, as well as the possible optimality of bound (3), remains to be addressed.

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Appendix

It must be shown that for arbitrary $d \geq 0$, $\pi_{\mu, \sigma^2}(d)$ defined in (1) in terms of $\mu = \mu_1 + \mu_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$ is maximized when $\rho = 1$. To this end, rewrite this bound as a function of $t = \sigma^2/\mu^2$, viz.

$$\pi(t, d) = f_d(t)1\{d \leq h(t)\} + g_d(t)1\{d > h(t)\},$$

where $h(t) = \mu(1 + t)/2$,

$$f_d(t) = \mu - \frac{d}{1 + t} \quad \text{and} \quad g_d(t) = \frac{\mu - d}{2} + \frac{1}{2} \mu \sqrt{\left(\frac{\mu - d}{\mu}\right)^2 + t}.$$ 

If $t_0 = (\sigma_1 - \sigma_2)^2/\mu^2$ and $t_1 = (\sigma_1 + \sigma_2)^2/\mu^2$, one must then show that one has $\pi(t, d) \leq \pi(t_1, d)$ for all $t \in [t_0, t_1]$ and any fixed retention amount $d \geq 0$. 

Three cases must be considered, according as $d \geq h(t_1)$, $d \leq h(t_0)$ or $h(t_0) < d < h(t_1)$. When $d \geq h(t_1)$, $\pi(t, d) = g_d(t)$ for all $t_0 \leq t \leq t_1$, and hence is an increasing function of $t$, whence $\pi(t, d) \leq \pi(t_1, d)$. Likewise when $d \leq h(t_0)$, $\pi(t, d) = f_d(t)$ for all $t_0 \leq t \leq t_1$ and since the latter is monotone increasing, $\pi(t, d) \leq \pi(t_1, d)$ once again.

Finally, suppose that $h(t_0) < d < h(t_1)$ and let $t^*_d = h^{-1}(d) = 2d/\mu - 1$. Two subcases must be considered, according as $t_0 \leq t \leq t^*_d$ or $t^*_d \leq t \leq t_1$. In the first subcase, $\pi(t, d) = g_d(t) \leq g_d(t^*_d) = f_d(t^*_d) = \mu/2$ since $g_d$ is monotone increasing. In the second subcase, $\pi(t, d) = f_d(t)$ and this increasing function is maximized at $t_1$, so that $\pi(t, d) \leq \pi(t_1, d) = f_d(t_1)$ for all $t_0 \leq t \leq t_1$. This completes the argument.

References


