On life insurance reserves in a stochastic mortality and interest rates environment

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Abstract

The calculation of the reserves in a stochastic mortality and interest rates environment for a general portfolio of life insurance policies is examined. The first two moments of the prospective loss random variable for the general portfolio are derived. A Monte Carlo simulation method is used to estimate the distribution of this random variable. Another approximation of the prospective loss random variable which is based on the assumption of a large portfolio is also considered. In the numerical examples, a discrete-time model for the stochastic interest rates is assumed. ©1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The objective in this paper is to examine the calculation of the reserves in a stochastic mortality and interest rates environment for a general portfolio of life insurance policies.

A first approach is to define a prospective loss random variable \( L_{PTF} \) for the portfolio by assuming stochastic mortality and interest rates models. The expectation of the prospective loss random variable \( L_{PTF} \) corresponds to the prospective reserves. A contingency provision can be added to the prospective reserves and the level of this provision is determined upon the basis of the information that one has on the distribution of this random variable. A first assessment of the risk of the portfolio is to determine the first two moments of the random variable \( L_{PTF} \). However, the knowledge of the distribution of this random variable provides a better assessment of the risk for the portfolio. A Monte Carlo method can be used to estimate this distribution.

When the portfolio contains a large number of policies, the mortality risk becomes less important in comparison to the interest risk. Consequently, another approach in the assessment of the risk for the portfolio is to approximate the prospective loss random variable \( L_{PTF} \) by a random variable \( M_{PTF} \) which is a prospective loss random variable of the portfolio defined by assuming deterministic mortality and stochastic interest rates. This approach is proposed, for instance, by Parker (1997) and Frees (1990). One purpose of the present study is to verify the quality of this approximation.
approximation. To that end, we compare the distributions (estimated by a Monte Carlo method) of the prospective loss random variables $L_{PTF}$ and $M_{PTF}$ for different compositions of life insurance portfolios.


The paper is organized as follows. After a brief description of the general portfolio of life insurance contracts, we present the assumptions regarding the decrements and the discounting function. Two expressions are given for the definition of the prospective loss random variable $L_{PTF}$ for the general portfolio. The first and second moments of $L_{PTF}$ are derived in both cases. Then, the prospective loss random variable $M_{PTF}$ is defined and its first and second moments are derived. These two random variables are compared and a Monte Carlo simulation method is applied for the estimation of their distributions. In the numerical examples, a discrete-time AR(1) model for the stochastic interest rates is assumed.

2. Description of the portfolio

At a given valuation date, consider a general portfolio with $m$ fully discrete life temporary and endowment insurance contracts. These contracts may have different terms $n_i$ with specific death benefits $b_i$ and pure dotation amounts $c_{i,i=1,\ldots,m}$. The age of the insured $i$ at the date of issue is $x_i$. The difference between issue and valuation dates is defined by $r_i$ (with $r_i < n_i$). The mortality of the insureds depends on their profile ($\tau_i$). The premium $\pi_i$ for the contract $i$ $(i=1,\ldots,m)$ is payable at the beginning of the year over the duration of the contract.

3. Assumptions

3.1. Decrements

Aiming at keeping the presentation simple, mortality is assumed to be the unique cause of decrement. The curtate lifetime of the insured $i$ is denoted by the random variable $K_i(i=1,\ldots,m)$. We assume that $K^* = \{K_i,i=1,2,\ldots,m\}$ is a sequence of independent random variables (not necessarily identically distributed). In the usual actuarial notation (see Gerber (1997), Bowers et al. (1997), DeVylder (1997)), we have

$$P(K_i = k) = k \cdot p^{(tau_i)}_{x_i+k} = k \cdot q^{(tau_i)}_{x_i},$$

and

$$P(K_i \geq k) = k \cdot p^{(tau_i)}_{x_i^+}, \quad k = 0,1,\ldots.$$ 

The exponent $(\tau_i)$ indicates that the choice of mortality table depends on the profile $(\tau_i)$ of the insured $i$.

3.2. Discount function

A discrete-time model for stochastic interest rates is assumed in the definition of the discount functions. Let $\delta(k)$ be the force of interest over the $k$th year $(k-1, k]$ for $k = 1, 2, \ldots$. A certain number of stochastic discrete-time models for $\delta^* = \{\delta(k), k = 1, 2, \ldots\}$ are presented in the actuarial literature. The independent and identically
distributed (i.i.d.) model is applied by Waters (1978), Panjer and Bellhouse (1980), Bellhouse and Panjer (1981), Dhaene (1989,1992), Frees (1990), Lai and Frees (1995) deal with autoregressive and moving average models such as the AR($k$), MA($m$), or the more general ARIMA($k,m$) models. Norberg (1994,1997b), propose a Markov Chain model. ARCH models are also considered by Lai and Frees (1995).

At time zero, the discount function of one unit payable at time $k$ is

$$v(k) = \exp(-I(k)), \quad k = 1, 2, \ldots,$$

where $I(k) = \sum_{j=1}^{k} \delta(j)$. We assume that $v(0) = 1$. The $u$th conditional moment of $v(k)$ given $\delta(0)$ is

$$E[v(k)^u|\delta(0)] = E \left[ \exp \left( -u \sum_{j=1}^{k} \delta(j) \right) | \delta(0) \right]. \quad (1)$$

The present value of cash flows of one unit payable at the beginning of each year during $k$ years is

$$\bar{a}(k) = 1 + \sum_{j=1}^{k-1} v(j).$$

The conditional expectations of $\bar{a}(k)$, $\bar{a}(k)^2$ and $v(k)\bar{a}(k)$ are given by

$$E[\bar{a}(k)|\delta(0)] = 1 + \sum_{i=1}^{k-1} E[v(i)|\delta(0)], \quad E[\bar{a}(k)^2|\delta(0)] = \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} E[v(i)v(j)|\delta(0)].$$

$$E[v(k)\bar{a}(k)|\delta(0)] = \sum_{i=0}^{k-1} E[v(i)v(k)|\delta(0)]. \quad (2)$$

Also, with $k_1 < k_2$, we have

$$E[\bar{a}(k_1)\bar{a}(k_2)|\delta(0)] = \sum_{i=0}^{k_1-1} \sum_{j=0}^{k_2-1} E[v(i)v(j)|\delta(0)] = E[\bar{a}(k_1)\bar{a}(k_2-1)|\delta(0)] + \sum_{i=0}^{k_1-1} E[v(i)v(k_2-1)|\delta(0)]. \quad (3)$$

The expectations in (1)–(3) are obtained by conditioning on $\delta(0)$. For simplicity, we write $E[v(k)^u]$ for $E[v(k)^u|\delta(0)]$.

The random variables $\{K_i, i = 1, 2, \ldots, m\}$ and $\{\delta(k), k = 1, 2, \ldots\}$ are assumed to be independent. No expenses are assumed.

### 4. Prospective loss random variables for the portfolio

For the general portfolio described in Section 2, two prospective loss random variables, $L^{\text{PTF}}$ and $M^{\text{PTF}}$, are considered. The random variable $L^{\text{PTF}}$ is defined in an environment with stochastic mortality and stochastic interest rates. Two different formulations of $L^{\text{PTF}}$ are presented. In the definition of the random variable $M^{\text{PTF}}$, we assume deterministic mortality and stochastic interest rates. The definition of $M^{\text{PTF}}$ is given in the next section. These two random variables are compared in Section 6. We will see that $M^{\text{PTF}}$ can be used as an approximation of $L^{\text{PTF}}$. 
4.1. First expression of $L^\text{PTF}$

4.1.1. Introduction and notations

Let $r_i L(K_i, \delta^*)$ be the prospective loss random variable for the contract $i$. It corresponds to the net difference at time $r_i$ between the present value of future benefits and the present value of future premiums from time $r_i$ to $n_i$. The random variable $r_i L(K_i, \delta^*)$ is defined by

$$r_i L(K_i, \delta^*) = \begin{cases} b_i v(K_i - r_i + 1) - \pi_i \bar{a}(K_i - r_i + 1), & K_i = r_i, r_i + 1, \ldots, n_i - 1 \\ c_i v(n_i - r_i) - \pi_i \bar{a}(n_i - r_i), & K_i = n_i, n_i + 1, \ldots \end{cases}$$

The random variable $r_i L(K_i, \delta^*)$ is dependent since they are all defined with the same discount functions (see Gerber (1997), Parker (1997), Frees (1990)).

The random variable $L^\text{PTF}$ can be expressed as the sum of the individual prospective loss random variables

$$L^\text{PTF} = \sum_{i=1}^{m} r_i L(K_i, \delta^*).$$

When it is clear from the context, we use the simplified form $r_i L_i$ rather than $r_i L(K_i, \delta^*)$.

4.1.2. Explicit formulas for $E[\tau_i L_i]$, $E[r_i L_i^2]$ and $E[r_i L_i, L_j]$

The expressions of $E[\tau_i L_i]$, $E[r_i L_i^2]$ and $E[r_i L_i, L_j]$ are derived by conditioning on the survival of the insureds of the portfolio to the valuation date. Since $K_i$ and $\delta^*$ are independent, the expression of $E[\tau_i L_i]$ is given by

$$E[\tau_i L_i] = \sum_{k=0}^{n_i-r_i-1} \left( b_i E[v(k+1)] - \pi_i E[\bar{a}(k+1)] \right) k q_{a+\tau_i}^{(r_i)} + \{c_i E[v(n_i-r_i)] - \pi_i E[\bar{a}(n_i-r_i)] \} n_i-r_i P_{a+\tau_i}^{(r_i)}.$$  (4)

Then, the expression derived for $E[r_i L_i^2]$ is

$$E[r_i L_i^2] = \sum_{k=0}^{n_i-r_i-1} \left\{ b_i^2 E[v(k+1)^2] - 2b_i \pi_i E[v(k+1) \bar{a}(k+1)] + \pi_i^2 E[\bar{a}(k+1)^2] \right\} k q_{a+\tau_i}^{(r_i)}$$

$$+ \{c_i^2 E[v(n_i-r_i)^2] - 2c_i \pi_i E[v(n_i-r_i) \bar{a}(n_i-r_i)] + \pi_i^2 E[\bar{a}(n_i-r_i)^2] \} n_i-r_i P_{a+\tau_i}^{(r_i)}.$$  (5)

After some calculations, the expression obtained for $E[r_i L_i, L_j]$ is

$$E[r_i L_i, L_j] = \sum_{k_j=0}^{n_j-r_j-1} \sum_{k_i=0}^{n_i-r_i-1} \left\{ b_i b_j E\left[v(k_i+1) v(k_j+1)\right] + \pi_i \pi_j E\left[\bar{a}(k_i+1) \bar{a}(k_j+1)\right] \right\} k_i q_{a+\tau_i}^{(r_i)} k_j q_{a+\tau_j}^{(r_j)}$$

$$+ \sum_{k_j=0}^{n_j-r_j-1} \left\{ b_i c_j E\left[v(k_i+1) v(n_j-r_j)\right] + \pi_i \pi_j E\left[\bar{a}(k_i+1) \bar{a}(n_j-r_j)\right] \right\} k_i q_{a+\tau_i}^{(r_i)} n_j-r_j P_{a+\tau_j}^{(r_j)}$$

$$+ \sum_{k_i=0}^{n_i-r_i-1} \left\{ -b_i \pi_j E\left[v(k_i+1) \bar{a}(n_j-r_j)\right] + \pi_i \pi_j E\left[\bar{a}(k_i+1) \bar{a}(n_j-r_j)\right] \right\} k_i q_{a+\tau_i}^{(r_i)} n_j-r_j P_{a+\tau_j}^{(r_j)}$$

$$+ \sum_{k_i=0}^{n_i-r_i-1} \left\{ -\pi_i c_j E\left[\bar{a}(k_i+1) v(n_j-r_j)\right] + \pi_i \pi_j E\left[\bar{a}(k_i+1) \bar{a}(n_j-r_j)\right] \right\} k_i q_{a+\tau_i}^{(r_i)} n_j-r_j P_{a+\tau_j}^{(r_j)}$$

$$- \sum_{k_i=0}^{n_i-r_i-1} \sum_{k_j=0}^{n_j-r_j-1} \left\{ \pi_i \pi_j E\left[\bar{a}(k_i+1) \bar{a}(k_j+1)\right] - \pi_i \pi_j E\left[\bar{a}(k_i+1) \bar{a}(n_j-r_j)\right] + \pi_i \pi_j E\left[\bar{a}(k_i+1) \bar{a}(k_j+1)\right] - \pi_i \pi_j E\left[\bar{a}(n_i-r_i) \bar{a}(n_j-r_j)\right] \right\} k_i q_{a+\tau_i}^{(r_i)} k_j q_{a+\tau_j}^{(r_j)}.$$
4.1.3. Explicit formulas for $E[L_{PTF}^r]$ and $\text{Var}[L_{PTF}^r]$

The expectation of $L_{PTF}^r$, the prospective loss random variable of the whole portfolio, is given by

$$E[L_{PTF}^r] = \sum_{i=1}^{m} E[r_i L_i],$$

(7)

where the expression of $E[r_i L_i]$ is given in Eq. (4). The total prospective reserve for the whole portfolio determined under the principle of equivalence corresponds to $E[L_{PTF}^r]$.

The variance of $L_{PTF}^r$ is determined with the following relation

$$\text{Var}[L_{PTF}^r] = \text{Var}\left[\sum_{i=1}^{m} r_i L_i\right] = \sum_{i=1}^{m} \sum_{j=1}^{m} \text{Cov}[r_i L_i, r_j L_j] = \sum_{i=1}^{m} \text{Var}[r_i L_i] + \sum_{i=1}^{m} \sum_{j=1, j \neq i}^{m} \text{Cov}[r_i L_i, r_j L_j].$$

(8)

where

$$\text{Var}[r_i L_i] = E[r_i L_i^2] - E[r_i L_i]^2,$$

$$\text{Cov}[r_i L_i, r_j L_j] = E[r_i L_i, r_j L_j] - E[r_i L_i] E[r_j L_j].$$

The expressions of $E[r_i L_i]$, $E[r_i L_i^2]$ and $E[r_i L_i, r_j L_j]$ are respectively given by (4)–(6).

Because of the dependence between $r_i L_i$ and $r_j L_j$ for $i \neq j$, $\text{Cov}[r_i L_i, r_j L_j]$ may not equal zero. The calculation of $\text{Var}[L_{PTF}^r]$ requires the calculations of $\text{Cov}[r_i L_i, r_j L_j]$ for all combinations of $i$ and $j$. The computation of $\text{Var}[L_{PTF}^r]$ thus becomes very problematic (even impracticable) for a large number of contracts (greater than 200, say). An alternative to this computation problem is to recursively calculate $\text{Var}[L_{PTF}^r]$ by using the following relation:

$$\text{Var}\left[\sum_{i=1}^{m} r_i L_i\right] = \text{Var}\left[\sum_{i=1}^{m-1} r_i L_i\right] + \text{Var}[r_m L_m] + 2 \sum_{i=1}^{m-1} \text{Cov}[r_m L_m, r_i L_i].$$

However, even if this recursive relation is applied, the computational problem remains. We can get around this problem if the definition of $L_{PTF}^r$ is expressed differently.

4.2. Second expression for $L_{PTF}^r$

4.2.1. Introduction and notations

Following Lai and Frees (1995), $L_{PTF}^r$ can also be expressed as the present value of the stochastic cash flows for the whole portfolio

$$L_{PTF}^r = \sum_{j=0}^{n} \text{CF}(j) v(j),$$
where  
\[ n = \max_{i \in \{1, \ldots, m\}} (n_i - k_i) \]
and \( CF(j) \) is the cash flows at time \( j \) representing the net difference between the benefits and the premiums paid at time \( j (j = 0, 1, \ldots, n) \).

The following notation is inspired from Parker (1997). In the definition of \( CF(j) \), we use the following random variables:

\[
D_{i,j} = \begin{cases} 
1, & \text{if the insured } i \text{ dies within the time interval } (j - 1, j] \\
0, & \text{otherwise,}
\end{cases}
\]

\[
S_{i,j} = \begin{cases} 
1, & \text{if the insured } i \text{ is alive at time } j \\
0, & \text{otherwise}
\end{cases}
\]

with

\[
E[D_{i,j}] = j^{-1}q_i^{(\tau_i)} r_i \\
E[S_{i,j}] = j p_i^{(\tau_i)} r_i \\
\text{Var}[D_{i,j}] = j^{-1}q_i^{(\tau_i)} (1 - j^{-1}q_i^{(\tau_i)}) \\
\text{Var}[S_{i,j}] = j p_i^{(\tau_i)} (1 - j p_i^{(\tau_i)}) 
\]

For \( k < j \), we have

\[
\text{Cov}[D_{i,k}, D_{i,j}] = -j^{-1}q_i^{(\tau_i)} R_{k-1}^{(\tau_i)} q_i^{(\tau_i)} \\
\text{Cov}[D_{i,j}, S_{i,j}] = -j^{-1}q_i^{(\tau_i)} p_i^{(\tau_i)} + j p_i^{(\tau_i)} \\
\text{Cov}[D_{i,k}, S_{i,j}] = -j^{-1}q_i^{(\tau_i)} R_{k-1}^{(\tau_i)} p_i^{(\tau_i)} \\
\text{Cov}[S_{i,k}, D_{i,j}] = j^{-1} P_j^{(\tau_j)} (1 - j^{-1}q_j^{(\tau_j)}) \\
\text{Cov}[S_{i,k}, S_{i,j}] = j p_j^{(\tau_j)} (1 - k p_j^{(\tau_j)}) 
\]

\[ R_{k-1}^{(\tau_i)} = P_i^{(\tau_i)} - j^{-1} q_i^{(\tau_i)} R_{k-1}^{(\tau_i)} \\
R_{k-1}^{(\tau_j)} = P_j^{(\tau_j)} - j^{-1} q_j^{(\tau_j)} R_{k-1}^{(\tau_j)} 
\]

\[ P_i^{(\tau_i)} = 1 - q_i^{(\tau_i)} \\
P_j^{(\tau_j)} = 1 - q_j^{(\tau_j)} \]

For \( k < j \), we have

\[
\text{Cov}[S_{i,k}, D_{i,j}] = j^{-1} q_j^{(\tau_j)} R_{k-1}^{(\tau_j)} P_j^{(\tau_j)} (1 - j^{-1}q_j^{(\tau_j)}) \\
\text{Cov}[S_{i,k}, S_{i,j}] = j p_j^{(\tau_j)} (1 - k p_j^{(\tau_j)}) 
\]

\[ R_{k-1}^{(\tau_i)} = P_i^{(\tau_i)} - j^{-1} q_i^{(\tau_i)} R_{k-1}^{(\tau_i)} \\
R_{k-1}^{(\tau_j)} = P_j^{(\tau_j)} - j^{-1} q_j^{(\tau_j)} R_{k-1}^{(\tau_j)} \]

Since the lifetimes of the insureds are independent, it is clear that the covariances in Eq. (10) are zero for a pair of variables from two different lifetimes. For the portfolio, the cash flow \( CF(j) \) at time \( j (j = 1, 2, \ldots, n) \) is given by

\[
CF(j) = \sum_{i=1}^{m} D_{i,j} b_i 1_{(n_i - r_i \geq j)} - \sum_{i=1}^{m} S_{i,j} \pi_i 1_{(n_i - r_i > j)} + \sum_{i=1}^{m} S_{i,n_i - r_i} \epsilon_i 1_{(n_i - r_i = j)},
\]

and at time 0, we have

\[
CF(0) = -\sum_{i=1}^{m} \pi_i 1_{(n_i - r_i > 0)}.
\]

The indicator function \( 1_A \) has a value of 1 if the condition \( A \) is fulfilled and a value of 0, otherwise. For \( j = 1, 2, \ldots, n \), the random variable \( CF(j) \) depends only on the mortality experience of the insureds. The random variables \( \{CF(1), CF(2), \ldots, CF(n)\} \) are dependent. Given the assumptions, \( \{CF(1), CF(2), \ldots, CF(n)\} \) and \( \{\delta(k), k = 1, 2, \ldots\} \) are independent. \( CF(0) \) is a constant. We also have

\[
\sum_{j=0}^{n} CF(j) v(j) = \sum_{i=1}^{m} L(K_i, \delta^*)\]
We need the explicit expressions of $E[CF(j)]$, $Var[CF(j)]$ and $Cov[CF(j), CF(k)]$ in order to derive the expressions of $E[L_{PTF}]$ and $Var[L_{PTF}]$.

4.2.2. Explicit formulas for $E[CF(j)]$, $Var[CF(j)]$ and $Cov[CF(j), CF(k)]$

After simple calculations and using Eq. (9), the expression obtained for $E[CF(j)]$ ($j = 1, 2, \ldots, n$) is

$$E[CF(j)] = \sum_{i=1}^{m} b_{i} \left( \frac{r_{i}}{d_{x_{i}+r_{i}}} \right) 1(\alpha_{i}-r_{i} \geq j) - \sum_{i=1}^{m} \pi_{i} j P_{x_{i}+r_{i}}^{(r_{i})} 1(\alpha_{i}-r_{i} > j) + \sum_{i=1}^{m} c_{i} n_{i}-r_{i} P_{x_{i}+r_{i}}^{(r_{i})} 1(n_{i}-r_{i} = j)$$

(12)

and

$$E[CF(0)] = CF(0) = -\sum_{i=1}^{m} \pi_{i}.$$  

Similarly, from Eqs. (9) and (10), the variance of $CF(j)$ (for $j = 1, 2, \ldots, n$) is given by

$$Var[CF(j)] = \sum_{i=1}^{m} b_{i}^{2} Var[D_{i,j}] 1(n_{i}-r_{i} \geq j) + \sum_{i=1}^{m} c_{i}^{2} Var[S_{i,n_{i}-r_{i}}] 1(n_{i}-r_{i} = j)$$

$$+ 2 \sum_{i=1}^{m} b_{i} c_{i} Cov[D_{i,n_{i}-r_{i}}, S_{i,n_{i}-r_{i}}] 1(n_{i}-r_{i} = j) + \sum_{i=1}^{m} \pi_{i}^{2} Var[S_{i,j}] 1(n_{i}-r_{i} > j)$$

$$+ 2 \sum_{i=1}^{m} \pi_{i} b_{i} Cov[D_{i,j}, S_{i,j}] 1(n_{i}-r_{i} > j).$$

(13)

Then, we get the following expression for $Cov[CF(k), CF(j)]$ (assuming $k < j$)

$$Cov[CF(k), CF(j)] = \sum_{i=1}^{m} b_{i}^{2} Cov[D_{i,k}, D_{i,j}] 1(n_{i}-r_{i} \geq j) + \sum_{i=1}^{m} b_{i} c_{i} Cov[D_{i,k}, S_{i,n_{i}-r_{i}}] 1(n_{i}-r_{i} = j)$$

$$+ \sum_{i=1}^{m} - b_{i} \pi_{i} Cov[D_{i,k}, S_{i,j}] 1(n_{i}-r_{i} > j) + \sum_{i=1}^{m} \pi_{i} \pi_{i} Cov[S_{i,k}, S_{i,j}] 1(n_{i}-r_{i} > j)$$

$$+ \sum_{i=1}^{m} - b_{i} \pi_{i} Cov[S_{i,k}, D_{i,j}] 1(n_{i}-r_{i} \geq j) + \sum_{i=1}^{m} \pi_{i} c_{i} Cov[S_{i,k}, S_{i,n_{i}-r_{i}}] 1(n_{i}-r_{i} = j).$$

(14)

4.2.3. Explicit formulas for $E[L_{PTF}]$ and $Var[L_{PTF}]$

The prospective loss random variable for the whole portfolio, $L_{PTF}$, is the sum of the present value of the future cash flows $CF(j)$ at the durations $j = 0, 1, \ldots, m$,

$$L_{PTF} = \sum_{j=0}^{n} CF(j) v(j).$$

Since $K^{*} = \{K_{i}, i = 1, \ldots, m\}$ and $\delta^{*} = \{\delta(k), k = 1, 2, \ldots\}$ are independent, the expectation of $L_{PTF}$ is given by

$$E[L_{PTF}] = \sum_{j=0}^{n} E[CF(j)] E[v(j)],$$

(15)

where the expression of $E[CF(j)]$ is given by Eq. (12). Clearly, Eq. (7) and (15) for $E[L_{PTF}]$ are equal.
Using Shiriayev (1996), the variance can be derived by conditioning either on \( K^* = \{ K_i, i = 1, \ldots, m \} \) or \( \delta^* = \{ \delta(k), k = 1, 2, \ldots \} \) (see also Parker (1997)). By conditioning on \( K^* = \{ K_i, i = 1, \ldots, m \} \), we obtain

\[
\text{Var} \left[ L^{\text{PTF}} \right] = E \left[ \text{Var} \left[ L^{\text{PTF}} \mid K^* \right] \right] + \text{Var} \left[ E \left[ L^{\text{PTF}} \mid K^* \right] \right],
\]

where

\[
E \left[ \text{Var} \left[ L^{\text{PTF}} \mid K^* \right] \right] = E \left[ \text{Var} \left[ \sum_{j=1}^{n} \text{CF}(j) \cdot v(j) \mid K^* \right] \right] = E \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} \text{CF}(j) \cdot \text{CF}(k) \cdot \text{Cov} [v(j), v(k)] \right] = \sum_{j=1}^{n} \sum_{k=1}^{n} E[\text{CF}(j) \cdot \text{CF}(k)] \cdot \text{Cov}[v(j), v(k)]
\]

and

\[
\text{Var} \left[ E \left[ L^{\text{PTF}} \mid K^* \right] \right] = \text{Var} \left[ E \left[ \sum_{j=1}^{n} \text{CF}(j) \cdot v(j) \mid K^* \right] \right] = \text{Var} \left[ \sum_{j=1}^{n} \text{CF}(j) \cdot E[v(j)] \right] = \sum_{j=1}^{n} \sum_{k=1}^{n} E[v(j)] \cdot E[v(k)] \cdot \text{Cov}[\text{CF}(j), \text{CF}(k)],
\]

where the expression of \( \text{Cov}[\text{CF}(j), \text{CF}(k)] \) is given by Eq. (14) for \( j \neq k \) and by Eq. (13) for \( j = k \).

By conditioning on \( \delta^* = \{ \delta(k), k = 1, 2, \ldots \} \), it follows that

\[
\text{Var} \left[ L^{\text{PTF}} \right] = E \left[ \text{Var} \left[ L^{\text{PTF}} \mid \delta^* \right] \right] + \text{Var} \left[ E \left[ L^{\text{PTF}} \mid \delta^* \right] \right]
\]

where

\[
E \left[ \text{Var} \left[ L^{\text{PTF}} \mid \delta^* \right] \right] = E \left[ \text{Var} \left[ \sum_{j=1}^{n} \text{CF}(j) \cdot v(j) \mid \delta^* \right] \right] = E \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} v(j) \cdot v(k) \cdot \text{Cov} [\text{CF}(j), \text{CF}(k)] \right] = \sum_{j=1}^{n} \sum_{k=1}^{n} E[v(j) \cdot v(k)] \cdot \text{Cov}[\text{CF}(j), \text{CF}(k)]
\]

and

\[
\text{Var} \left[ E \left[ L^{\text{PTF}} \mid \delta^* \right] \right] = \text{Var} \left[ E \left[ \sum_{j=1}^{n} \text{CF}(j) \cdot v(j) \mid \delta^* \right] \right] = \text{Var} \left[ \sum_{j=1}^{n} v(j) \cdot E[\text{CF}(j)] \right] = \sum_{j=1}^{n} \sum_{k=1}^{n} E[\text{CF}(j)] \cdot E[\text{CF}(k)] \cdot \text{Cov}[v(j), v(k)].
\]

By comparing Eq. (16) and (19) with Eq. (8), we can see that the calculation of \( \text{Var} \left[ L^{\text{PTF}} \right] \) is much easier under the second expression of \( L^{\text{PTF}} \) than under its first expression. The calculation is feasible for any number of contracts. The second formulation of \( L^{\text{PTF}} \) also has the advantage of being very flexible since it is possible to adapt it to every type of contracts with elaborated characteristics. The alternative expression of \( L^{\text{PTF}} \) in terms of stochastic cash flows leads to the definition of the prospective loss random variable \( M^{\text{PTF}} \).
5. Definition of $M_{\text{PTF}}$

When the number of contracts in a portfolio is large, the mortality risk tends to take less importance in comparison to the interest risk. This is why, when a large portfolio is assumed, the distribution of the random variable $L_{\text{PTF}}$ can be approximated by the distribution of the random variable $M_{\text{PTF}}$ which is the present value of the expected future cash flows. The latter random variable is defined as follows:

$$M_{\text{PTF}} = \sum_{j=0}^{n} E[\text{CF}(j)]v(j),$$

where the expression of $E[\text{CF}(j)]$ is given by Eq. (12). Using expected future cash flows is equivalent to applying the assumption of deterministic mortality. The random variable $M_{\text{PTF}}$ can be considered as the present value of an annuity where the annual payments correspond to the expected future cash flows. Therefore, the expectation and the variance of $M_{\text{PTF}}$ are derived as follows:

$$E[M_{\text{PTF}}] = \sum_{j=0}^{n} E[\text{CF}(j)]E[v(j)],$$

and

$$\text{Var}[M_{\text{PTF}}] = \sum_{j=1}^{n} E[\text{CF}(j)]^2 \text{Var}[v(j)] + 2 \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} E[\text{CF}(j)]E[\text{CF}(k)] \text{Cov}[v(j), v(k)].$$

Comparing Eq. (15) and (22), it is clear that $E[L_{\text{PTF}}]$ and $E[M_{\text{PTF}}]$ are equal. The variance of $M_{\text{PTF}}$ underestimates the variance of $L_{\text{PTF}}$.

6. Comparison of $L_{\text{PTF}}$ and $M_{\text{PTF}}$

The prospective life reserves for the general portfolio correspond to the expectation of either $L_{\text{PTF}}$ or $M_{\text{PTF}}$. A contingency provision can be added to the prospective reserves in regards to the risk of the portfolio. The sensitivity of the reserves to the assumptions is also important. The level of the contingency provision is determined upon the basis of the information that one has on the distribution of the random variable $L_{\text{PTF}}$. It follows that a better knowledge of the distribution of $L_{\text{PTF}}$ is required. A first assessment of the risk associated to the portfolio is made with the variance of $L_{\text{PTF}}$. Following Frees (1990), Norberg (1997a,b), the inequality of Chebyshev (see Shiriaev (1996)) with the relations (15) and (16) (or 19) can be applied in order to obtain an upper bound for the deviation from the mean. Because of the limitations of this approach, Monte Carlo simulations are the usual method to estimate the distribution of $L_{\text{PTF}}$. In this regard, one needs to simulate both the realizations of the force of interest for each year and the time of death of the insureds of the portfolio. However, the Monte Carlo method has the disadvantage of being computationally intensive for a large portfolio.

An alternative approach is to approximate $L_{\text{PTF}}$ by $M_{\text{PTF}}$ when the number of contracts in the portfolio is large enough. This approximation reduces the time spent for the simulation by eliminating the simulation of the lifetimes of each insured. This approximation was also proposed by Frees (1990), Parker (1997). We have seen above that $M_{\text{PTF}}$ may represent the present value of an annuity with variable payments. The numerical evaluation of its distribution was studied among others by Dufresne (1990), De Schepper and Goovaerts (1992), De Schepper et al. (1994), Parker (1994b,1997), Vanneste et al. (1997), Milevsky and Posner (1998).

We compare the distributions of $L_{\text{PTF}}$ and $M_{\text{PTF}}$ for different portfolios in Section 8.
7. Stochastic models of interest rates

As it was mentioned in Section 3, a certain number of models for stochastic interest rates were proposed in the actuarial literature. The theory presented in previous sections can be used with any of these models. In our numerical tests, we have assumed a discrete-time AR(1) model which is frequently proposed in the actuarial literature. Our choice was simply motivated by its convenience since it is possible to derive the analytic expressions of $E[v_i], \text{Var}[v_i]$ and $\text{Cov}[v_i, v_j]$. It is also known (see Lai and Frees (1995) for instance) that the AR(1) model is the discrete counterpart of the continuous time Ornstein-Uhlenbeck process. Additional motivations for autoregressive models are given in Frees (1990).

Under the AR(1) model, the force of interest at time $k$, $\delta(k)$, is defined by the following recursive relation:

$$\delta(k) - \delta = \phi(\delta(k) - \delta) + \sigma \varepsilon(k), \quad \text{for } k = 1, 2, \ldots,$$

where $\{\varepsilon(k)\}$ is an i.i.d. sequence of Gaussian random variables with 0 mean and unit variance. The sequence is covariance–stationary if $|\phi| < 1$. It is known that the stationary distribution of $\delta(k)$ is normal:

$$\delta(k) \sim \text{Norm}(E[\delta(k)], \text{Var}[\delta(k)]),$$

with

$$E[\delta(k)] = \delta, \quad \text{Var}[\delta(k)] = \frac{\sigma^2}{1 - \phi^2}, \quad \text{Cov}[\delta(k), \delta(k+m)] = \frac{\sigma^2}{1 - \phi^2} \phi^m, \quad m = 0, 1, \ldots.$$

The conditional distribution of $\delta(k)$ given $\delta(0)$ is also normal with the following moments:

$$E[\delta(k)|\delta(0)] = \delta + (\delta(0) - \delta) \phi^k, \quad \text{Var}[\delta(k)|\delta(0)] = \frac{\sigma^2}{1 - \phi^2} (1 - \phi^{2k}),$$

$$\text{Cov}[\delta(k), \delta(k+m)|\delta(0)] = \frac{\sigma^2}{1 - \phi^2} \phi^m (1 - \phi^{2k}).$$

Defining $I(k) = \sum_{j=1}^{k} \delta(j)$, for $k = 1, 2, \ldots$, we have that $v(k) = \exp(-I(k))$. For $k = 1, 2, \ldots$, the conditional distribution of $I(k)$ is normal

$$I(k) \sim \text{Norm}(E[I(k)|\delta(0)], \text{Var}[\delta(k)|\delta(0)]),$$

where

$$E[I(k)|\delta(0)] = E \left[ \sum_{j=1}^{k} \delta(j)|\delta(0) \right] = k\delta + (\delta(0) - \delta) \phi \left( 1 - \frac{\phi^k}{1 - \phi} \right),$$

and

$$\text{Var}[I(k)|\delta(0)] = \text{Var} \left[ \sum_{j=1}^{k} \delta(j)|\delta(0) \right]$$

$$= \sum_{j=1}^{k} \text{Var}[\delta(j)|\delta(0)] + 2 \sum_{j=1}^{k-1} \sum_{i=j+1}^{k} \text{Cov}[\delta(j), \delta(i)|\delta(0)] = \frac{\sigma^2}{1 - \phi^2} \left( k - \phi^2 \frac{1 - \phi^{2k}}{1 - \phi^2} \right)$$

$$+ 2 \frac{\sigma^2}{1 - \phi^2} \phi \left( k - 1 - \phi^2 \frac{1 - \phi^{2(k-1)}}{1 - \phi^2} \right) - \phi \frac{1 - \phi^{k-1}}{1 - \phi} + \phi^k \frac{1 - \phi^{k-1}}{1 - \phi}.$$
Table 1
Portfolio A of \(m\) temporary life insurance contracts\(^a\)
\[
\begin{array}{cccc|cccc}
\phi & \sigma & E[L_{PTF}^m], Var[L_{PTF}^m] & E[M_{PTF}^m], Var[M_{PTF}^m] \\
& & m = 270 & m = 2700 & m = 27,000 & m = 270,000 \\
0.5 & 0.01 & 0.974 & 0.974 & 0.974 & 0.974 & 0.974 \\
& & 0.553 & 0.058 & 0.099 & 0.004 & 0.003 \\
0.5 & 0.02 & 0.980 & 0.980 & 0.980 & 0.980 & 0.980 \\
& & 0.568 & 0.069 & 0.019 & 0.014 & 0.014 \\
0.5 & 0.03 & 0.988 & 0.988 & 0.988 & 0.988 & 0.988 \\
& & 0.594 & 0.088 & 0.037 & 0.032 & 0.031 \\
0.9 & 0.01 & 0.983 & 0.983 & 0.983 & 0.983 & 0.983 \\
& & 0.576 & 0.076 & 0.026 & 0.021 & 0.020 \\
0.9 & 0.02 & 1.016 & 1.016 & 1.016 & 1.016 & 1.016 \\
& & 0.671 & 0.147 & 0.095 & 0.090 & 0.089 \\
0.9 & 0.03 & 1.075 & 1.075 & 1.075 & 1.075 & 1.075 \\
& & 0.872 & 0.302 & 0.245 & 0.239 & 0.239 \\
\end{array}
\]
\(^a\)Other parameters of the AR(1) model: \(\delta = 0.08, \delta_0 = 0.08.\)

Table 2
Portfolio B of \(m\) endowment life insurance contracts\(^a\)
\[
\begin{array}{cccc|cccc}
\phi & \sigma & E[L_{PTF}^m], Var[L_{PTF}^m] & E[M_{PTF}^m], Var[M_{PTF}^m] \\
& & m = 270 & m = 2700 & m = 27,000 & m = 270,000 \\
0.5 & 0.01 & 17.931 & 17.931 & 17.931 & 17.931 \\
& & 1.503 & 1.366 & 1.352 & 1.350 & 1.350 \\
0.5 & 0.02 & 18.041 & 18.041 & 18.041 & 18.041 \\
& & 5.635 & 5.497 & 5.483 & 5.481 & 5.481 \\
0.5 & 0.03 & 18.227 & 18.227 & 18.227 & 18.227 \\
0.9 & 0.01 & 18.143 & 18.143 & 18.143 & 18.143 \\
0.9 & 0.02 & 18.915 & 18.915 & 18.915 & 18.915 \\
& & 40.498 & 40.362 & 40.349 & 40.347 & 40.347 \\
0.9 & 0.03 & 20.296 & 20.296 & 20.296 & 20.296 \\
& & 112.946 & 112.809 & 112.793 & 112.793 & 112.793 \\
\end{array}
\]
\(^a\)Other parameters of the AR(1) model: \(\delta = 0.08, \delta_0 = 0.08.\)

Table 3
Portfolio C of \(m\) temporary and endowment life insurance contracts\(^a\)
\[
\begin{array}{cccc|cccc}
\phi & \sigma & E[L_{PTF}^m], Var[L_{PTF}^m] & E[M_{PTF}^m], Var[M_{PTF}^m] \\
& & m = 270 & m = 2700 & m = 27,000 & m = 270,000 \\
0.5 & 0.01 & 9.453 & 9.453 & 9.453 & 9.453 \\
& & 0.723 & 0.407 & 0.375 & 0.372 & 0.372 \\
0.5 & 0.02 & 9.510 & 9.510 & 9.510 & 9.510 \\
& & 1.862 & 1.544 & 1.512 & 1.508 & 1.508 \\
0.5 & 0.03 & 9.608 & 9.608 & 9.608 & 9.608 \\
& & 3.836 & 3.513 & 3.481 & 3.478 & 3.478 \\
0.9 & 0.01 & 9.563 & 9.563 & 9.563 & 9.563 \\
& & 2.786 & 2.477 & 2.445 & 2.442 & 2.442 \\
0.9 & 0.02 & 9.966 & 9.966 & 9.966 & 9.966 \\
0.9 & 0.03 & 10.685 & 10.685 & 10.685 & 10.685 \\
& & 31.223 & 30.869 & 30.834 & 30.830 & 30.830 \\
\end{array}
\]
\(^a\)Other parameters of the AR(1) model: \(\delta = 0.08, \delta_0 = 0.08.\)
Table 4
Portfolio A of \(m\) temporary life insurance contracts

<table>
<thead>
<tr>
<th>(\phi)</th>
<th>(\sigma)</th>
<th>(L_{90}, L_{95}) (m = 270)</th>
<th>(m = 2700)</th>
<th>(m = 27000)</th>
<th>(M_{90}, M_{95})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.01</td>
<td>1.932</td>
<td>1.287</td>
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<td>1.051</td>
</tr>
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<td>1.953</td>
<td>1.322</td>
<td>1.159</td>
<td>1.135</td>
</tr>
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<td>1.380</td>
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<td>23.010</td>
<td>22.958</td>
<td>22.958</td>
<td>22.970</td>
</tr>
<tr>
<td>0.9</td>
<td>0.01</td>
<td>1.962</td>
<td>1.342</td>
<td>1.195</td>
<td>1.172</td>
</tr>
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<td></td>
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<td>22.211</td>
<td>22.145</td>
<td>22.145</td>
<td>22.144</td>
</tr>
<tr>
<td>0.9</td>
<td>0.02</td>
<td>2.073</td>
<td>1.525</td>
<td>1.430</td>
<td>1.410</td>
</tr>
<tr>
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<td></td>
<td>27.582</td>
<td>27.389</td>
<td>27.389</td>
<td>27.288</td>
</tr>
<tr>
<td>0.9</td>
<td>0.03</td>
<td>2.265</td>
<td>1.794</td>
<td>1.724</td>
<td>1.699</td>
</tr>
<tr>
<td></td>
<td></td>
<td>34.274</td>
<td>33.947</td>
<td>33.964</td>
<td>33.752</td>
</tr>
</tbody>
</table>

\(a\) Other parameters of the AR(1) model: \(\delta = 0.08, \delta_0 = 0.08.\)

Table 5
Portfolio B of \(m\) endowment life insurance contracts

<table>
<thead>
<tr>
<th>(\phi)</th>
<th>(\sigma)</th>
<th>(L_{90}, L_{95}) (m = 270)</th>
<th>(m = 2700)</th>
<th>(m = 27000)</th>
<th>(M_{90}, M_{95})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
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<td>19.540</td>
<td>19.473</td>
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<td>0.5</td>
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<td>21.144</td>
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<td>21.144</td>
</tr>
<tr>
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<td></td>
<td>22.265</td>
<td>22.123</td>
<td>22.058</td>
<td>22.074</td>
</tr>
<tr>
<td>0.5</td>
<td>0.03</td>
<td>23.010</td>
<td>22.958</td>
<td>22.958</td>
<td>22.970</td>
</tr>
<tr>
<td></td>
<td></td>
<td>24.762</td>
<td>24.548</td>
<td>24.447</td>
<td>24.481</td>
</tr>
<tr>
<td>0.9</td>
<td>0.01</td>
<td>22.211</td>
<td>22.145</td>
<td>22.145</td>
<td>22.144</td>
</tr>
<tr>
<td></td>
<td></td>
<td>23.681</td>
<td>23.527</td>
<td>23.384</td>
<td>23.410</td>
</tr>
<tr>
<td>0.9</td>
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<td>27.582</td>
<td>27.389</td>
<td>27.389</td>
<td>27.288</td>
</tr>
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<td></td>
<td>31.062</td>
<td>30.862</td>
<td>30.525</td>
<td>30.598</td>
</tr>
<tr>
<td>0.9</td>
<td>0.03</td>
<td>34.274</td>
<td>33.947</td>
<td>33.964</td>
<td>33.752</td>
</tr>
<tr>
<td></td>
<td></td>
<td>40.849</td>
<td>40.705</td>
<td>40.013</td>
<td>40.104</td>
</tr>
</tbody>
</table>

\(a\) Other parameters of the AR(1) model: \(\delta = 0.08, \delta_0 = 0.08.\)

Table 6
Portfolio C of \(m\) temporary and endowment life insurance contracts

<table>
<thead>
<tr>
<th>(\phi)</th>
<th>(\sigma)</th>
<th>(L_{90}, L_{95}) (m = 270)</th>
<th>(m = 2700)</th>
<th>(m = 27000)</th>
<th>(M_{90}, M_{95})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.01</td>
<td>10.560</td>
<td>11.155</td>
<td>10.258</td>
<td>10.253</td>
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<td></td>
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<td>10.876</td>
<td>11.667</td>
<td>10.481</td>
<td>10.476</td>
</tr>
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<td>0.5</td>
<td>0.02</td>
<td>11.338</td>
<td>11.155</td>
<td>11.145</td>
<td>11.141</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11.870</td>
<td>11.667</td>
<td>11.617</td>
<td>11.628</td>
</tr>
<tr>
<td>0.5</td>
<td>0.03</td>
<td>12.250</td>
<td>12.111</td>
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<td>12.099</td>
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<td>13.132</td>
<td>12.937</td>
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<td>12.885</td>
</tr>
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<td>11.847</td>
<td>11.682</td>
<td>11.664</td>
<td>11.631</td>
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<td>12.375</td>
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<td>18.025</td>
<td>17.802</td>
<td>17.832</td>
<td>17.743</td>
</tr>
</tbody>
</table>

\(a\) Other parameters of the AR(1) model: \(\delta = 0.08, \delta_0 = 0.08.\)
Also, the expression of Cov\[I(k), I(k + m)|\delta(0)\]\] is given by

\[
\text{Cov}[I(k), I(k + m)|\delta(0)] = \text{Cov}\left[I(k), I(k) + \sum_{i=k+1}^{k+m} \delta(i)|\delta(0)\right]
\]

\[
= \text{Cov}[I(k), I(k)|\delta(0)] + \text{Cov}\left[I(k), \sum_{i=k+1}^{k+m} \delta(i)|\delta(0)\right]
\]

\[
= \text{Var}[I(k)|\delta(0)] + \frac{\sigma^2}{1 - \phi^2} \frac{\phi(1 - \phi^m)}{(1 - \phi)} \sum_{j=1}^{k} (1 - \phi^{2j})
\]

\[
= \text{Var}[I(k)|\delta(0)] + \frac{\sigma^2}{1 - \phi^2} \frac{\phi(1 - \phi^m)}{(1 - \phi)} \left(k - \frac{\phi^2(1 - \phi^{2(k-1)})}{(1 - \phi^2)}\right).
\]

The conditional distribution of \(v(k) = \exp(-I(k))\) is lognormal. It follows that the expressions of \(E[v(k)^n|\delta(0)]\) and \(E[v(k)v(k + m)|\delta(0)]\) are

\[
E[v(k)^n|\delta(0)] = \exp\left(-nE[I(k)|\delta(0)] + \frac{n^2\text{Var}[I(k)|\delta(0)]}{2}\right),
\]

and

\[
E[v(k)v(k + m)|\delta(0)] = \exp(-A + B),
\]
Fig. 2. Simulated density of $L^{PTF}/m$ and $M^{PTF}/m$. Portfolio of endowment insurance contracts.

with

$$A = E[I(k)|\delta(0)] + E[I(k+m)|\delta(0)]$$

$$B = \frac{1}{2} [\text{Var}[I(k)|\delta(0)] + 2\text{Cov}[I(k), I(k+m)|\delta(0)] + \text{Var}[I(k+m)|\delta(0)]].$$

8. Numerical examples and observations

8.1. Numerical examples

In the numerical examples, we use three comparable portfolios. Portfolio A is formed of $m$ temporary life insurance contracts while portfolio B is formed of $m$ endowment life insurance contracts. Portfolio C is constituted of halves, one from each of portfolios A and B. The descriptions of the three portfolios are given in the Appendix A. The compositions of portfolios A and B (ages of insureds, benefit amounts, etc.) are identical except that the contracts in A and B are respectively temporary and endowment life insurance.

We compare the distributions of $L^{PTF}/m$ and $M^{PTF}/m$ where

- $L^{PTF}/m$ = average prospective losses by contract assuming stochastic cash flows;
- $M^{PTF}/m$ = average prospective losses by contract assuming expected cash flows.

We use these measures

1. $\text{Var}[L^{PTF}/m]$ and $\text{Var}[M^{PTF}/m]$;
2. $L_{\alpha}^{PTF}$ such as $P(L^{PTF}/m \leq L_{\alpha}^{PTF}) = \alpha$;
3. $M_{\alpha}^{PTF}$ such as $P(M^{PTF}/m \leq M_{\alpha}^{PTF}) = \alpha$. 
Fig. 3. Simulated density of $L_{PTF}^m$ and $M_{PTF}^m$. Portfolio of temporary and endowment insurance contracts.

The exact values of $\text{Var}[L_{PTF}^m]$ and $\text{Var}[M_{PTF}^m]$ are calculated with Eq. (16), (19) and (23). The values of $L_\alpha$ and $M_\alpha$ are obtained by Monte Carlo simulations.

The parameters of the AR(1) model are given in the tables. We use the mortality table CA-80/82 with adjustment factors 1.00 and 0.80 for $i_D^1$ (male) and $i_D^3$ (female). For $i_D^1$ and $i_D^3$, we have $q_x^{(1)} = 1 - (1 - q_x)^{1.00}$ and $q_x^{(3)} = 1 - (1 - q_x)^{0.80}$ where $q_x$ is taken from the mortality table CA-80/82.

8.2. Remarks and observations

In Tables 1–3, we present the values of $E[L_{PTF}^m]$, $\text{Var}[L_{PTF}^m]$, $E[M_{PTF}^m]$, $\text{Var}[M_{PTF}^m]$ and in Tables 4–6, we present the values of $L_{90}$, $L_{95}$, $M_{90}$ and $M_{95}$ for the three portfolios with different parameters $\phi$ and $\sigma$ in the AR(1) model. The parameters $\delta$ and $\delta_0$ are the same in the six tables. As expected, $E[L_{PTF}^m]$ and $E[M_{PTF}^m]$ are equal. For the three portfolios, we observe the convergence of $\text{Var}[L_{PTF}^m]$, $L_{90}$, $L_{95}$ toward $\text{Var}[M_{PTF}^m]$, $M_{90}$ and $M_{95}$, respectively. However, convergence speed differs between the three portfolios. We notice that convergence is less rapid for portfolio A (temporary life insurance contracts) than for portfolios B and C. For the portfolio of temporary life insurance contracts, the impact of the mortality element seems to be very important and it should not be underestimated in comparison with the impact of the interest element (a similar observation is made in Norberg, 1997a). For the portfolio of endowment life contracts, the convergence is very rapid. Interesting also is the behavior of the convergence of $\text{Var}[L_{PTF}^m]$, $L_{90}$, $L_{95}$ for the portfolio C which is similar to the one for the portfolio B.

The results also provide an insight on the sensitivity of the values of $E[L_{PTF}^m]$, $\text{Var}[L_{PTF}^m]$, $E[M_{PTF}^m]$, $\text{Var}[M_{PTF}^m]$, $L_{90}$, $L_{95}$, $M_{90}$ and $M_{95}$ to the choice of parameters in the interest rate model. All these values increase when the parameters $\phi$ or $\sigma$ increase. The effect of the variation of the parameters $\phi$ or $\sigma$ is more important on
Fig. 4. Simulated density of $L^{PTF}_m$ and $M^{PTF}_m$. Portfolio of temporary insurance contracts.

Fig. 5. Simulated density of $L^{PTF}_m$ and $M^{PTF}_m$. Portfolio of endowment insurance contracts.
Fig. 6. Simulated density of $L_{PTF}$ and $M_{PTF}$. Portfolio of temporary and endowment insurance contracts.

Var[$L_{PTF}$/$m$], Var[$M_{PTF}$/$m$], $L_{90}$, $L_{95}$, $M_{90}$ and $M_{95}$ than on $E[L_{PTF}$/$m$] and $E[M_{PTF}$/$m$] for the three portfolios. However, the magnitude of the increase in the values of Var[$L_{PTF}$/$m$], Var[$M_{PTF}$/$m$], $L_{90}$, $L_{95}$, $M_{90}$ and $M_{95}$ is less important for portfolio A (especially when $m$ is low) than for portfolios B and C.

In Figs. 1–6, we exhibit the curves of the simulated (probability) densities of $L_{PTF}$/$m$ (for $m$=270, 2700, 27000) and $M_{PTF}$/$m$ for the three portfolios and for $\phi = 0.5$ (Figs. 1–3) and $\phi = 0.9$ (Figs. 4–6). We observe that the influence of the parameter $\phi$ is more important on the portfolios B and C. Also, the curves are almost overlapping in Figs. 2,3,5 and 6. It is not the case however in Figs. 1 and 4.

From these numerical results, it seems that the approximation of the distribution of $L_{PTF}$ by the distribution of $M_{PTF}$ works well for the portfolio B of endowment life contracts and for the portfolio C of endowment and temporary life insurance contracts even for medium size portfolios. For portfolio A of temporary life contracts (only), the approximation must be applied with care and the size of the portfolio must be very large because of the importance of the mortality element.

9. Conclusion

We have studied the determination of the reserves for a portfolio of life insurance contracts in the context of stochastic mortality and interest rates. Two formulations were proposed for the definition of the prospective loss random variables $L_{PTF}$. The first two moments within these two formulations were derived. An approximation of $L_{PTF}$ by the random variable $M_{PTF}$ was considered. We examined its quality in the numerical examples.
Acknowledgements

This research was funded in part by an operating grant from the Natural Sciences and Engineering Research Council of Canada and by a grant from the Chaire en Assurance L’Industrielle-Alliance (Université Laval). The authors are grateful to Mrs. Véronique Bouchard and Mr. Yann Lussier for their technical support. The authors would also like to thank Pr. Ragnar Norberg and an anonymous referee for their valuable comments.

Appendix A. Portfolio C

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References


