On the discrete-time compound renewal risk model with dependence

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A B S T R A C T
In this paper, we study the discrete-time renewal risk model with dependence between the claim amount random variable and the interclaim time random variable. We consider several dependence structures between the claim amount random variable and the interclaim time random variable. Recursive formulas are derived for the probability mass function and the moments of the total claim amount over a fixed period of time. In the context of ruin theory, explicit expressions for the expected penalty (Gerber–Shiu) function are derived for special cases. We also discuss how the discrete-time compound renewal risk model with dependence can be used to approximate the corresponding continuous time compound renewal risk model with dependence. Numerical examples are provided to illustrate different topics discussed in the paper.

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1. Introduction

In ruin theory, the compound renewal risk model and its special case, the compound binomial risk model, are based on the assumption of independence between the claim amount random variable (r.v.) and the interclaim r.v. In practical contexts, this assumption can be restrictive. For example, in modelling natural catastrophic events, we can expect that, on the occurrence of a catastrophe, the total claim amount (or the intensity of the catastrophe) and the time elapsed since the previous catastrophe are dependent (see e.g. Nikoloulopoulos and Karlis (2008)). The compound binomial model was first proposed by Gerber (1988a,b) and further examined e.g. by Shiu (1989), Willmot (1993), Dickson (1994, 2005), and Cheng et al. (2000). The discrete time renewal risk model has been studied in the actuarial literature by Pavlova and Willmot (2004), Li (2005a,b) and Cossette et al. (2006) among others. In the last decade, contributions such as Yuen and Guo (2001) and Cossette et al. (2003, 2004a,b,c) have considered temporal dependence in the compound binomial risk model.

In the present paper, we consider the discrete-time compound renewal risk model assuming a general dependence structure between the interclaim time r.v. and the claim amount r.v. This paper treats the modelling of dependence in risk theory and shows that interesting results can be derived even if a dependence relation is assumed between the interclaim time r.v. and the claim amount r.v.

The paper is structured as follows. In the next section, we present the risk model and examples of dependence structures between the interclaim time r.v. and the claim amount r.v. We examine the properties of the aggregate claim amount process in Section 3. In Section 4, we define the surplus process and the usual ruin measures. A recursive expression is given for the finite-time ruin probabilities. The behavior of the adjustment coefficient in function of the dependence relation is examined. In Section 5, we derive explicit expressions for the expected Gerber–Shiu function in two special cases. In the final section, we discuss how to use the discrete-time renewal risk model with dependence to approximate the corresponding continuous time renewal risk model with dependence. Numerical examples are provided to illustrate different topics discussed in the paper.

2. Model and dependence structures

In the discrete time renewal risk model with dependence, the claim number process \( N = \{N_k, k \in \mathbb{N}^+\} \) is a renewal process with interclaim times \( \{W_j, j \in \mathbb{N}^+\} \) and \( W_0 = 0 \) where \( \{W_j, j \in \mathbb{N}^+\} \) is a sequence of independent and strictly positive integer-valued r.v.’s distributed as the canonical r.v. \( W \). The r.v. \( W \) has a probability mass function (p.m.f.) \( f_W \), a cumulative distribution function (c.d.f.) \( F_W \) and a probability generating function (p.g.f.) \( f_W^* \). The claim amount r.v.’s \( \{X_j, j \in \mathbb{N}^+\} \), where \( X_j \) corresponds to the amount of the \( j \)th claim, are assumed to be a sequence of independent and strictly positive integer-valued r.v.’s distributed as the canonical r.v. \( X \) with p.m.f. \( f_X \), c.d.f. \( F_X \) and p.g.f. \( f_X^* \).

We assume that \( \{(X_j, W_j), j \in \mathbb{N}^+\} \) forms a sequence of i.i.d. random vectors distributed as the canonical r.v. \( (X, W) \), in which the components may be dependent. The joint p.m.f. and the joint c.d.f. of \( (X, W) \) is denoted by \( f_{X,W}(x, t) \) and \( F_{X,W}(x, t) \) with...
As explained in Marshall and Olkin (1985) and Hawkes (1972), the probabilities \( p_{00}, p_{11}, p_{01}, p_{10} \) correspond to the values of the joint mass probability function of the couple \((I, J)\) with a bivariate Bernoulli distribution, that is \( \Pr(I = i, J = j) = p_{ij} \), \( \Pr(I = i) = p_{i+} \), and \( \Pr(J = j) = p_{+j} \), for \( i, j \in \{0, 1\} \). Letting \( \tau = \text{Cov}(I, J) = p_{11} - p_{1+}p_{+1} \), Pearson’s correlation coefficient \( r_{p}(I, J) \) for \((I, J)\) is equal to \( \frac{\tau}{\sqrt{p_{1+}p_{+1} - p_{11}}} \) and it takes values between \(-1\) and \(1\). According to Marshall and Olkin (1985), Pearson’s correlation coefficient for \((X, W)\) is given by

\[
r_p(X, W) = \frac{p_{1+}p_{+1} - p_{11}}{\sqrt{p_{1+}p_{+1} - p_{11}}}.
\]

with \(-\frac{1}{\sqrt{2}} \leq r_p(X, W) \leq 1\). This distribution is also discussed in Basu and Dhar (1995) and Johnson et al. (1997) with a different parametrization.

2.1.3. Bivariate geometric (Type 3)

The bivariate geometric distribution of type 3 is obtained by a procedure proposed by Marshall and Olkin (1995). Suppose that \((Z_1, Z_2)\) has a bivariate exponential distribution. If we define \(X = [Z_1] + 1\) and \(W = [Z_2] + 1\), \((X, W)\) has a bivariate geometric distribution. Applying this procedure with bivariate exponential distributions (see e.g. in chapter 47 of Kotz et al. (2002) for examples of bivariate exponential distributions), we can create a large class of bivariate geometric distributions.

We examine an example of this procedure. We suppose that \((Z_1, Z_2)\) follows a bivariate exponential distribution suggested by Gumbel (see Model 2 in chapter 47 of Kotz et al. (2002)) with the following joint c.d.f.

\[
F_{Z_1, Z_2}(z_1, z_2) = (1 - e^{-\beta_1z_1})(1 - e^{-\beta_2z_2}) + \alpha(1 - e^{-\beta_1z_1})(1 - e^{-\beta_2z_2})(e^{-\beta_1z_1})(e^{-\beta_2z_2}),
\]

for \(z_1 \geq 0, z_2 \geq 0, \) and \(-1 \leq \alpha \leq 1\). For \(X = [Z_1] + 1\) and \(W = [Z_2] + 1\) and letting \(e^{-\beta_1} = 1 - q_1 = p_1\), we have the following expression for \(f_{X, W}(x, t)\),

\[
f_{X, W}(1, 1) = 1 - p_1 - p_2 + (1 + \alpha)p_1p_2 - \alpha p_2^2p_2 - \alpha p_1p_2^2 + \alpha p_1^2p_2
\]

for \(x = 1\) and \(t = 1\):

\[
f_{X, W}(x, 1) = p_1^{(x-1)}q_1 + (1 + \alpha)p_1^{(x-1)}q_1p_2 - \alpha p_1^{(x-1)}(1 - p_1)p_2
\]

for \(x = 2, 3, \ldots\); and \(t = 1\):

\[
f_{X, W}(1, t) = p_2^{(t-1)}q_2 + (1 + \alpha)p_2^{(t-1)}q_2p_1 - \alpha p_2^{(t-1)}(1 - p_2)p_1
\]

for \(x = 1\) and \(t = 2, 3, \ldots\).

The p.g.f. of \((X, W)\) is given by

\[
\hat{f}_{X, W}(s_1, s_2) = s_1s_2f_{X, W}(1, 1) + \sum_{x=2}^{\infty} s_1^xs_2^{2x}f_{X, W}(x, t)
\]

\[
+ \sum_{t=2}^{\infty} s_1^xs_2^{2t}f_{X, W}(x, t) + \sum_{x=2}^{\infty} \sum_{t=x}^{\infty} s_1^xs_2^{2t}f_{X, W}(x, t).
\]
= s_1 s_2 (1 - p_1) (1 - p_2) + \alpha t (1 - p_1) (1 - p_2) p_1 p_2 \\
+ \frac{s_1^2}{s_2} \frac{p_1 q_1}{1 - s_1 p_1} + (1 + \alpha) s_1 s_2 \frac{p_1 q_2}{1 - s_1 p_1} + \frac{s_1^2}{s_2} \frac{p_2 q_1}{1 - s_2 p_2} + (1 + \alpha) s_1 s_2 \frac{p_2 q_2}{1 - s_2 p_2} \\
- \alpha s_1^2 \frac{p_1^2 (1 - p_2) p_1}{1 - s_1 p_1^2} - \alpha s_1 s_2 \frac{p_1 q_1 p_2}{1 - s_1 p_1} + \alpha s_1^2 \frac{p_2^2 (1 - p_1) p_2}{1 - s_2 p_2} + (1 + \alpha) s_1 s_2 \frac{p_2 q_2 p_1}{1 - s_2 p_2} \\
+ \frac{s_1^2}{s_2} \frac{(1 - p_1 - p_2 + p_1 p_2)}{(1 - s_1 p_1)(1 - s_2 p_2)} - \alpha s_1^2 \frac{(1 - p_1^2 - p_2^2)}{(1 - s_1 p_1^2)(1 - s_2 p_2)} \\
- \alpha s_1^2 \frac{(1 - p_1 - p_2^2 + p_1^2 p_2^2)}{(1 - s_1 p_1)(1 - s_2 p_2)} + \alpha \frac{s_1^2}{s_2} \frac{(1 - p_1^2 - p_2^2 + \alpha p_1 p_2^2)}{(1 - s_1 p_1^2)(1 - s_2 p_2^2)}. \\
(7)

2.2. Structures based on copulas

A copula $C$ is the distribution function of a bivariate distribution with uniform marginals. By the theorem of Sklar (see e.g. Nelsen (2006)) any bivariate distribution function $F$ with marginals $F_1$ and $F_2$ can be written as $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$, for some copula $C$. This copula is unique if $F$ is continuous. Otherwise it is only unique on the range of the marginals. For a detailed treatment on the theory of copulas, we refer to Nelsen (2006) or Joe (1997).

Assume a bivariate random vector $(U, V)$ with continuous uniform marginals and with a dependence structure defined by a copula $F(u, v) = C(u, v)$ with $(u, v) \in [0, 1] \times [0, 1]$. We mention copulas that we use in this paper: Independence copula with $C^I(u, v) = uv$; comonotonic copula with $C^c(u, v) = \min(u, v)$; countermonotonic copula with $C^c(u, v) = \max(u + v - 1, 0)$; Frank copula with

$$c^F_\alpha(u, v) = -\frac{1}{\alpha} \ln \left\{ 1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{(e^{-\alpha} - 1)} \right\},$$

$\alpha \in \mathbb{R} \setminus \{0\}$ where $c^F_0 = C^I$; and Farlie–Gumbel–Morgenstern (FGM) copula with

$$c^{FGM}_{\alpha, \beta}(u_1, u_2) = u_1 u_2 + \alpha u_1 u_2 (1 - u_1) (1 - u_2),$$

$-1 \leq \alpha, \beta \leq 1$, where $c^{FGM}_0 = C^I$. It is important to mention that all copulas satisfy the inequalities $C^{-}(u, v) \leq C(u, v) \leq C^+(u, v)$, for $(u, v) \in [0, 1] \times [0, 1]$. The Frank copula belongs to the class of Archimedean copulas. An important property of the Frank copula is that it allows negative and positive dependence. The Frank copula is also comprehensive since it also includes the independence copula, the comonotonic, and the countermonotonic copulas as limit cases.

Copulas can be used to construct discrete bivariate distributions as suggested e.g. in Joe (1997) and as illustrated e.g. in Denuit et al. (2002), Cosssette et al. (2002), and Genest et al. (2003). This type of structure allows the coupling of various marginals. Let the bivariate distribution function $F_{X,W}(X, W)$ with marginals $F_X$ and $F_W$ be defined as

$$F_{X,W}(t, x) = F(C(t), F_W(x))$$

(9) for $(t, x) \in N^+ \times N^+$. The joint p.m.f. of $(X, W)$ is given by

$$f_{X,W}(t, x) = F_{X,W}(t, x + 1) - F_{X,W}(t - 1, x) - F_{X,W}(t, x - 1) + F_{X,W}(t - 1, x - 1),$$

(10) for $(x, t) \in N^+ \times N^+$ and where $F_{X,W}(x, t) = 0$ if $x = 0$ or $t = 0$. The p.g.f. is given by (1). As mentioned in Nelsen (2006), $F_{X,W}(t, x) = C(F_1(t), F_2(x))$ is defined on the support of $(X, W)$. A given discrete bivariate distribution does not lead to a unique copula. Genest and Nešlehová (2007) provide an excellent review about copulas linking discrete distributions. In the conclusion of their paper, they mention that dependence modelling with copulas as in (9) is a valid (and even attractive) approach for constructing bivariate distributions. Many stochastic dependence properties of a copula are inherited by the bivariate model obtained in (9). For example, stochastic ordering relations are preserved. However, care should be taken when one considers the parameter estimation of a copula in a discrete setting. Since the estimation of these parameters is not discussed in the present paper, we refer the interested readers to Genest and Nešlehová (2007) for further detail.

Latent (or threshold) models, as suggested in Joe (1997), Frey and McNeil (2003) and McNeil et al. (2005), can also be considered to construct discrete bivariate distributions using copulas. For example, assume a bivariate random vector $(U, V)$ with continuous uniform marginals and with a dependence structure defined by a copula $C$. Then, the bivariate r.v. $(X, W)$ with

$$X = \left\lfloor -\frac{1}{\beta} \ln (1 - U) \right\rfloor + 1$$

and

$$W = \left\lfloor -\frac{1}{\alpha} \ln (1 - V) \right\rfloor + 1$$

has a bivariate geometric distribution with parameters $q_1 = 1 - e^{-\alpha}$ and $q_2 = 1 - e^{-\beta}$ i.e. $f_X(x) = q_1 (1 - q_1)^{x-1}, x \in N^+$, and

$$f_W(t) = q_2 (1 - q_2)^{t-1}, t \in N^+.$$
As an example of this structure, let us suppose that the r.v. \( X \) is equal to \( \sum_{i=1}^{N} B_i \) with \( X = 0 \), if \( M = 0 \) and where \( B_1, B_2, \ldots \) form a sequence of i.i.d. r.v.'s (distributed as the canonical r.v. \( B \) with p.g.f. \( f_B \)) independent of \( M \). Given that \( \theta = \theta \) assume that \( M|\theta = \theta \) follows a Poisson distribution with mean \( \theta \lambda \), \( \lambda > 0 \). It implies that \( X|\theta = \theta \) has a compound Poisson distribution with p.g.f. \( E[s_{\theta}^X|\theta = \theta] = e^{\theta f_B(s-1)} \). Then, the expression for \( \hat{f}_{X,W}(s_1, s_2) \) is given by

\[
\hat{f}_{X,W}(s_1, s_2) = \sum_{t=1}^{\infty} \sum_{s \in \mathbb{A}_t} e^{\theta f_B(s-1)} f_{W, \theta}(t, \theta)
\]

Also, we have

\[
E[X] = E_{\theta}[E[X|\theta]] = E_{\theta}[E[\theta \lambda E[B]]] = \lambda E[\theta E[B]]
\]

and \( f_{X}(s) = f_{\theta}(e^{\theta f_B(s-1)} - 1) \). Moreover, if the distribution of \( (W, \theta) \) is defined with the FGM copula, if \( \theta \sim \text{Geom}(q_1) \) and if \( W \sim \text{Geom}(q_2) \), we obtain with (11) and (7)

\[
\hat{f}_{X,W}(s_1, s_2)
\]

3.1. Recursive formulas

In the next proposition, a recursive algorithm is provided to compute the p.m.f. of \( S_k \).

**Proposition 1.** In the discrete time renewal risk model with dependence, the p.m.f. of \( S_k, f_{S_k, x} \), for \( k \in \mathbb{N}^+ \) is given by

\[
f_{S_k}(j) = \sum_{k=1}^{\infty} f_{X,W}(x, t) f_{S_{k-1}}(j - x), \quad j \in \mathbb{N}^+.
\]

where \( f_{S_0}(i) = 1_{\{i=0\}} \).

**Proof.** We condition on \( (X_1, W_1) \) and the result follows from the stationarity of the aggregate claim amount process at renewal epochs. \( \Box \)

In our risk model, the presence of a dependence relation between \( W \) and \( X \) implies that the equality \( E[S_k] = E[N_k E[X]] \) is no longer valid. We can obviously determine the \( m \)th moment of \( S_k \) with \( E[S_k^m] = \sum_{j=0}^{\infty} j^m f_{S_k}(j) \). We can also derive recursive expressions to directly compute the \( m \)th moment of \( S_k \). In these expressions, we use the notation \( E[X^m; t] = \sum_{t=1}^{\infty} f_{X,W}(x, t) x^m \), for \( m \in \mathbb{N}^+ \). Since \( S_0 = 0, E[S_0^m] = 0 \), for \( m \in \mathbb{N}^+ \). The first moment can be computed with the following recursive expression:

\[
E[S_k] = \sum_{t=1}^{\infty} \sum_{x=1}^{\infty} f_{X,W}(x, t) E[X + S_{k-1}]
\]

\[
= \sum_{t=1}^{\infty} E[X; t] + \sum_{t=1}^{\infty} f_{W}(t) E[S_{k-1}].
\]

The procedure in (14) can be adapted to obtain the following recursive expression for the \( m \)th moment:

\[
E[S_k^m] = \sum_{t=1}^{\infty} \sum_{x=1}^{\infty} f_{X,W}(x, t) E\left[ \sum_{j=0}^{m} j E[X^m; t] \right]
\]

\[
= \sum_{t=1}^{\infty} E[X^m; t] + \sum_{t=1}^{\infty} f_{W}(t) E[S_{k-1}^m]
\]

\[
+ \sum_{j=1}^{m-1} \sum_{t=1}^{\infty} (j) \sum_{x=1}^{\infty} E[X^j; t] E[S_{k-1}^{m-j}].
\]

We consider the following example.

**Example 2 (Bivariate Geometric with Frank Copula).** We assume that \( (X, W) \) has a bivariate geometric distribution which is defined with a Frank copula (dependence parameter \( \alpha \)). The r.v. \( X \) (\( W \)) has a marginal geometric distribution with parameter \( q_1 = 1/10 \) \( (q_2 = 1/17) \) such that \( E[X] = 1/q_1 = 10 \) \( (E[W] = 1/q_2 = 12.5) \). In the following table, we provide the values of the covariance and Pearson’s correlation coefficient between \( W \) and \( X \) for the cases where \( \alpha = -10, -5, 5 \) and 10:

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Cov(( X, W ))</th>
<th>( r(( X, W )) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>-39.92243</td>
<td>-0.58234</td>
</tr>
<tr>
<td>-5</td>
<td>-31.17800</td>
<td>-0.49308</td>
</tr>
<tr>
<td>5</td>
<td>36.74107</td>
<td>0.51753</td>
</tr>
<tr>
<td>10</td>
<td>51.56878</td>
<td>0.72639</td>
</tr>
</tbody>
</table>

In Tables 1 and 2, we provide the values of the c.d.f. \( F_{\alpha}(x) \) and stop-loss premium \( \pi_{\alpha}(x) = E[\max(S_{\alpha} - x; 0)] \) for \( \alpha = -10, -5, 0 \) (independence), 5 and 10. The superscript \( \alpha \) refers to the dependence parameter of the Frank copula.
3.2. Present value of the claim amounts over a fixed period

In certain contexts (e.g., premium calculation and reinsurance), it can also be important to examine the behavior of the present value of the claim amounts over a fixed period. Let \( \delta \) be the force of interest. The present value at time \( t \) of an amount of \( 1 \) paid at time \( t \) corresponds to \( v^t = e^{-\delta t} \), \( t \geq 0 \).

The process for the present value of the claim amounts is represented by \( Z = \{Z_k, k \in \mathbb{N}\} \) with \( Z_0 = 0 \) and \( Z_k = \sum_{i=1}^{N_k} W_{i, k}^{+} + \sum_{i=1}^{N_k} W_{i, k}^{-} X_i \). The moments of \( Z_k \) can be computed recursively. The recursive expressions for the first moment and the \( m \)th moment are similarly obtained as in (14) and (15). We have

\[
E[Z_k] = \sum_{t=1}^{k} v^t E[X; t] + \sum_{t=1}^{k} f_W(t) v^t E[Z_{k-t}]
\]

and

\[
E[Z_k^m] = \sum_{t=1}^{k} v^{t \times m} E[X^{m}; t] + \sum_{t=1}^{k} v^{t \times m} f_W(t) E[Z_{k-t}^m] + \sum_{j=1}^{m-1} \binom{m}{j} \sum_{t=1}^{k} v^{t \times m} E[X^{j}; t] E[Z_{k-t}^{m-j}],
\]

where \( Z_0 = 0 \) and therefore, \( E[Z_0^m] = 0 \) for \( m \in \mathbb{N}^+ \).

4. Surplus process and ruin measures

4.1. Preliminaries

We define the surplus process \( U = \{U_k, k \in \mathbb{N}\} \) as \( U_0 = u \) and \( U_k = u + c \cdot S_k \) for \( k \in \mathbb{N}^+ \) where \( u \) is the initial surplus level and \( c \) is the level premium received per period. In this paper, we assume that \( c = 1 \) and that ruin occurs when the surplus falls...
below zero. The r.v. \( \tau = \inf_{k \geq 0} \{ k, U_k < 0 \} \) is defined as the time of ruin associated to \( U \) with \( \tau = \infty \) if \( U_k \geq 0 \) for all \( k \in \mathbb{N} \) (i.e., ruin does not occur).

Let \( \psi(u, n) = E[1_{[0,n]}] \) and \( \psi(u) = E[1_{[0,\infty)}] \) be the finite-time and infinite-time ruin probabilities where the indicator function \( 1_A = 1 \) if \( A \) is true and 0 otherwise. Their complements, the finite-time and infinite-time non-ruin probabilities, are respectively denoted by \( \phi(u, n) \) and \( \phi(u) \). To ensure that \( \psi(u) \) goes to 0 as \( u \to \infty \), the premium rate satisfies the usual condition

\[
\varepsilon E[|W|] > E[X].
\]

The premium rate \( \varepsilon = 1 \) includes a strictly positive risk margin \( \eta \) defined as \( \eta = E[X] / |\varepsilon E[X]| - 1 > 0 \).

In the following proposition, we provide a recursive algorithm to compute the finite-time non-ruin probabilities.

**Proposition 3.** In the discrete time renewal risk model with dependence, the finite-time non-ruin probabilities can be computed with

\[
\phi(u, n) = \sum_{j=1}^{n} \sum_{i=1}^{u-j} f_{X,W}(i,j) \phi(u-j-i, n-j) + F_W(n),
\]

for \( u \in \mathbb{N}, n \in \mathbb{N}^+ \) and with \( \phi(y, 0) = 1_{\{y \in \mathbb{N}\}} \).

**Proof.** We condition on both r.v.'s \( W_t \) and \( X_t \) and from the stationarity of the surplus process at renewal epochs, the result follows. \( \square \)

Let \( U_{t-1} \) and \( |U_t| \) be the surplus before ruin and the deficit at ruin. We define \( v = e^{-\delta} \in (0, 1) \) as a discount factor over one period and \( w(x,y) \) as a non-negative penalty function. Let \( \xi(u) = E[e^{\delta t} U_{t-1}, U_1], 1_{[0,\infty)}[U_0 = 0] \) be the expected discounted function introduced by Gerber and Shiu (1998). When \( \delta = 0 \) and \( w(x,y) = 1 \), \( \xi(u) \) corresponds to the probability of ruin \( \psi(u) \). If \( \delta = 0 \) and for \( w(x,y) = x, w(x,y) = y \) and \( w(x,y) = x+y, \xi(u) \) represents the expected value of the surplus before ruin (given that ruin occurs), the expected value of the deficit at ruin (given that ruin occurs), and the expected value of the claim amount causing ruin respectively. In Section 5, we derive explicit expressions for \( \xi(u) \) when \( \delta = 0 \) in some special cases.

### 4.2. Lundberg upper bound and impact of dependence

Let us define the sequence of i.i.d. r.v.'s \( Y_k, k \in \mathbb{N}^+ \) where \( Y_k = X_k - W_k \) for \( k \in \mathbb{N}^+ \). The r.v.'s \( Y_k \) are distributed as the canonical r.v. \( Y \) which takes value in \( \mathbb{Z} \). The p.m.f. of the r.v. \( Y \) is given by

\[
f_Y(y) = \sum_{t = \max(1, 1-y)}^{\infty} f_{X,W}(y + t, t),
\]

for \( y \in \mathbb{Z} \). We also define the risk process \( Y = \{ Y_k, k \in \mathbb{N} \} \), where \( V_k = \sum_{i=1}^{k} Y_i \) for \( k \in \mathbb{N} \) and \( V_0 = 0 \). Since \( E[V] = E[X] - \varepsilon E[W] < 0 \) due to the net profit condition in (16), \( V \) is a random walk with a negative drift. It is well-known (see e.g. Rolski et al. (1999)) using standard martingale arguments that the sequence \( \{\rho^k, k \in \mathbb{N}\} \) is an exponential martingale with \( \rho \) a real strictly positive solution to

\[
\hat{f}_Y(s) = E[s^X] = E[s^{X-W}] = 1,
\]

where

\[
\hat{f}_Y(s) = E[s^X] E[s^{-W}] = \sum_{t=1}^{\infty} \sum_{s=1}^{\infty} s^{s-1} f_{X,W}(x, t).
\]

The equality in (19) corresponds to the so-called Lundberg equation.

The expression in (20) depends on the dependence structure for \((X, W)\). The solution (if it exists) to (19), which is strictly greater than 1, is denoted \( \rho \) and is called the adjustment coefficient. Using martingale techniques (or an inductive approach), we obtain the Lundberg upper bound for the ruin probability i.e. \( \psi(u) \leq (\frac{\rho}{1-\rho})^u \) for \( u \in \mathbb{N} \). The adjustment coefficient is often considered as a “measure” of dangerousness associated to the risk process (see e.g. Gerber (1979) or Rolski et al. (1999)). It means that the degree of dangerousness decreases as the adjustment coefficient increases.

In order to examine the impact of the dependence relation on the adjustment coefficient, we briefly recall some basic definitions from stochastic ordering (see e.g. Shaked and Shanthikumar (1994), Müller and Stoyan (2002) or Denneit et al. (2005) for further detail). The univariate r.v.'s \( Z \) and \( Z' \) are ordered in increasing convex order (denoted \( Z \leq_{icx} Z' \)) if \( E[g(Z)] \leq E[g(Z')] \) holds for all increasing convex functions \( g \), such that the expectation exists. For example, \( g(z) = \max(z - x; 0) \) is an increasing convex function. Let \( Z = (Z_1, Z_2) \) and \( Z' = (Z'_1, Z'_2) \) be two bivariate random vectors with the same marginals. We say that \( Z \) is smaller than \( Z' \) in the concordance order (denoted \( Z \leq_{icx} Z' \)) if \( P(Z_1 \leq z_1, Z_2 \leq z_2) \geq P(Z'_1 \leq z_1, Z'_2 \leq z_2) \) for all \( z_1, z_2 \).

Let \((X, W)\) and \((X', W')\) be two bivariate random vectors. In Theorem 4 of Müller (2000), it is shown that \((X', W') \leq_{icx} (X', W')\) implies that \( X' - W' \leq_{icx} X - W \). Since the function \( g(u) = u^s \) is an increasing convex function, it follows that

\[
E[s^{(X'-W')} \leq E[s^{(X-W)}],
\]

for all \( s \geq 1 \). The inequality in (21) implies that \( \rho \leq \rho' \), where \( \rho \) and \( \rho' \) are the adjustment coefficients (if they exist) corresponding to the risk models defined with \((X, W)\) and \((X', W')\) respectively.

Ablercher and Teugels (2006) prove the same result in a continuous time renewal risk model with dependence, but they use a different approach which cannot be adapted in the discrete time risk model with dependence. The approach used above is more general and is valid in both models.

In the next three examples, we derive the expression of \( \hat{f}_Y(s) \).

#### 4.2.1. Example — Bivariate geometric (type 1)

When \((X, W)\) has a bivariate geometric distribution of type 1, with \( f_{X,W} \) and \( f_{X,W} \) given by (2) and (3) respectively, (20) becomes

\[
\hat{f}_Y(s) = \hat{f}_{X,W}(s, s^{-1}) = \frac{q_1 q_2}{q_1 q_2 - p_1 q_2 (s - 1) - p_2 q_1 (s^{-1} - 1)},
\]

Substituting (22) in (19), we find that the adjustment coefficient is equal to \( \rho = \frac{p_0 q_2}{p_2 q_1} \).

#### 4.2.2. Example — Bivariate geometric (type 2)

We assume that \((X, W)\) has a bivariate geometric distribution of type 2 where \( f_{X,W} \) and \( f_{X,W} \) are given by (4) and (5) respectively. It follows that

\[
f_Y(k) = \begin{cases} \frac{1}{1 - p_0}, & k = -1, 2, \ldots \\ \frac{1}{1 - p_0}, & k = 0 \\ \frac{1}{1 - p_0}, & k = 1, 2, \ldots \\ \frac{1}{1 - p_0}, & k = 0 \\ \frac{1}{1 - p_0}, & k = 1, 2, \ldots \\ \frac{1}{1 - p_0}, & k = 0 \\ \frac{1}{1 - p_0}, & k = 1, 2, \ldots \\ \end{cases}
\]

and

\[
\hat{f}_Y(s) = \frac{1}{1 - p_0} \left( \frac{p_0 p_{10}}{s - p_{10}} + p_{11} + \frac{p_0 p_{10}}{p_{10} + 1 - sp_{10}} \right).
\]
The Lundberg equation (19) becomes

\[ \tilde{f}_y(s) = \frac{1}{1 - p_{00}} \left( \frac{p_{01} p_{1+} - p_{0+} - p_{11} + p_{01} p_{1+}}{s - p_{0+}} + \frac{s p_{0+}}{1 - s p_{0+}} \right) \]

Rearranging (24) leads to

\[ 1 - p_{00} = \left( \left( \frac{p_{01} p_{1+} - p_{0+} - p_{11} + p_{01} p_{1+}}{s - p_{0+}} + \frac{s p_{0+}}{1 - s p_{0+}} \right) \right)^{-1} = 0. \] (25)

and also

\[ ((1 - p_{00})p_{1+} - \tau)^2 + ((p_{11} + p_{1+}p_{0+} - p_{00} \tau) \]

\[ - (1 - p_{00})(1 - p_{0+}p_{1+})z + ((1 - p_{00})p_{1+} - \tau) = 0. \] (25)

Eq. (25) has two solutions, one is equal to 1 and the other is the adjustment coefficient \( \rho > 1 \).

4.2.3. Example — Bivariate geometric (type 3)

We assume that \((X, W)\) has a bivariate geometric distribution of type 3 where \(f_{X,W}\) is given by (5). The Lundberg equation (19) becomes

\[ \tilde{f}_y(s) = \left( (1 - p_{11})(1 - p_{22}) + \alpha s (1 - p_{11}) (1 - p_{22}) p_{12} \right) \]

\[ + \frac{p_{11} q_{12}}{1 - s p_{11}} + (1 + \alpha) \frac{p_{22} q_{12} p_{11}}{1 - s p_{22}} \]

\[ - \alpha s \frac{p_{11}^2 (1 - p_{22})}{1 - s p_{11}} - \alpha s \frac{p_{22}^2}{1 - s p_{22}} + \alpha^2 \frac{p_{22}^2 (1 - p_{22})}{1 - s p_{22}} \]

\[ + \frac{s p_{11}}{s - p_{11}} + (1 + \alpha) \frac{s p_{22}}{s - p_{22}} \]

\[ - \alpha s \frac{p_{11}^2}{s - p_{11}} - \alpha s \frac{p_{22}^2}{s - p_{22}} + \alpha^2 \frac{p_{22}^2}{s - p_{22}} \]

\[ + (1 + \alpha) \frac{p_{11} - p_{22} + p_{11} p_{22}}{(1 - s p_{11})(s - p_{11})} - \alpha s \frac{p_{11}^2 - p_{22}^2}{(1 - s p_{11})(s - p_{11})} \]

\[ - \alpha s \frac{p_{11} - p_{22} + p_{11} p_{22}}{(1 - s p_{11})(s - p_{11})} + \alpha s \frac{p_{11}^2 - p_{22}^2}{(1 - s p_{11})(s - p_{11})} = 1. \] (26)

Rearranging (26) leads to a polynomial equation of order four.

4.2.4. Example — Bivariate geometric defined with the Frank copula

In this example, we want to illustrate the impact of dependence on the adjustment coefficient \( \rho \). We assume that \((X, W)\) has a bivariate geometric distribution which is defined with the Frank copula (dependence parameter = \( \alpha \)). The r.v. \(X\) and \(W\) have marginal geometric distributions with parameter \(q_1\) and \(q_2\), respectively. We also suppose the following constraint on the parameters \(q_1\) and \(q_2\)

\[ 1 - (1 - q_1) - (1 - q_2) = q_1 + q_2 - 1 > 0. \] (27)

The premium rate \(c\) is equal to 1. Given (27) and using (17) and (20), we obtain

\[ f_y(k) = \begin{cases} q_2 (1 - q_2)^k, & k = 1, -2, \ldots \\ 1 - (1 - q_1) - (1 - q_2), & k = 0 \\ q_1 (1 - q_1)^k, & k = 1, 2, \ldots \end{cases} \] (28)

and

\[ \tilde{f}_y(s) = \left( \frac{q_2 (1 - q_2)}{s - (1 - q_2)} + 1 - (1 - q_1) - (1 - q_2) + \frac{s q_1 (1 - q_1)}{1 - s (1 - q_1)} \right). \] (29)

Substituting (29) in (19) and letting \( p_1 = 1 - q_1 \), the Lundberg equation becomes

\[ \left( \frac{q_2 p_2}{s - p_2} + 1 - p_1 - p_2 + \frac{s q_1 p_1}{1 - s p_1} \right) = 1. \] (30)

Rearranging (30), we obtain

\[ q_2 p_2 (1 - s p_1) + s q_1 p_1 (s - p_2) - (p_1 + p_2) (1 - s p_1) (s - p_2) = 0. \] (31)

As in example with the bivariate geometric distribution (type 2), (31) has two solutions: 1 and \( \rho > 1 \).

5. Infinite-time ruin measures

In this section, we consider the computation of \( \zeta(u) \) when \( \delta = 0 \) in two specific cases. To simplify the notation, we drop the subscript \( \delta \) in \( \zeta(u) \), since \( \delta = 0 \). In the first case, we assume that \((X, W)\) follows a bivariate geometric distribution of type 2. In the second case, we suppose that \(X\) and \(W\) are defined on finite supports. In Section 5.3, we provide an explicit expression for \( \zeta(u) \) for the model described in Section 4.2.5.

5.1. Bivariate geometric (type 2)

We assume that \((X, W)\) follows a bivariate geometric distribution of type 2. For \( u = 0, 1, 2, \ldots \), we have

\[ \zeta(u) = \sum_{k=-\infty}^{u} \zeta(u-k) f_y(k) + o(u), \] (32)

where

\[ o(u) = \sum_{k=0}^{\infty} \sum_{t=1}^{\infty} f_{X,W}(t + k, t) w(t + u - 1, k - u) \]

\[ = \sum_{k=0}^{\infty} \sum_{t=1}^{\infty} \rho_{00}^{t-1} p_{01}^{t} \rho_{01}^{k-1} p_{1+} w(t + u - 1, k - u). \] (33)
From (33), we derive below the expressions for \( \omega(u) \) when
\[ w(x,y) = 1, \quad w(x,y) = x, \quad w(x,y) = y \text{ and } w(x,y) = x + y \]
\( \omega(u) \) represents the probability of ruin, the expected value of the surplus before ruin (given that ruin occurs), the expected value of the deficit at ruin (given that ruin occurs), and the expected value of the claim amount causing ruin respectively:

- **\( w(x,y) = 1 \):**
  \[
  \omega(u) = 1 - F_y(u) = \frac{p_{01}}{(1 - p_{00})p_{0+}}; \quad (34)
  \]

- **\( w(x,y) = x \):**
  \[
  \omega(u) = \sum_{k=0}^{\infty} \sum_{t=1}^{\infty} p_0^{-1} p_0 p_{0+}^{-1} p_{1+}(u + t - 1)
  = u \sum_{k=0}^{\infty} \sum_{t=1}^{\infty} p_0^{-1} p_0 p_{0+}^{-1} p_{1+}
  + \sum_{k=0}^{\infty} p_0^{-1} p_0 p_{0+}^{-1} p_{1+}(t - 1)
  = \frac{p_{01}p_{0+}}{(1 - p_{00})p_{0+}} + \frac{p_{01}p_{0+}}{(1 - p_{00})}
  = \frac{p_{01}p_{0+}}{(1 - p_{00})(1 - p_{0+})}; \quad (35)
  \]

- **\( w(x,y) = y \):**
  \[
  \omega(u) = \sum_{k=0}^{\infty} \sum_{t=1}^{\infty} p_0^{-1} p_0 p_{0+}^{-1} p_{1+}(k - u)
  = p_{01}p_{0+} + \sum_{k=1}^{\infty} p_0^{-1} p_0 p_{0+}^{-1}(k - u)
  = \frac{p_{01}p_{0+}}{(1 - p_{00})(1 - p_{0+})}; \quad (36)
  \]

- **\( w(x,y) = x + y \):**
  \[
  \omega(u) = u \frac{p_{01}p_{0+}}{(1 - p_{00})} + \frac{p_{01}p_{0+}}{(1 - p_{00})(1 - p_{0+})}
  = \frac{p_{01}p_{0+}}{(1 - p_{00})} \left( u + \frac{1}{1 - p_{00}} + \frac{1}{1 - p_{0+}} \right). \quad (37)
  \]

From (34)-(37), we clearly see the impact of the dependence on the values of \( w(x,y) \).

Using (23), (32) becomes

\[
\zeta(u) = \sum_{k=-\infty}^{1} \left( \zeta(u-k) f_y(k) + \zeta(u) f_y(0) \right)
+ \sum_{k=1}^{u} \zeta(u-k) f_y(k) + o(u)
= \frac{1}{1 - p_{00}} \left( \frac{p_{01}p_{0+}}{p_{0+}} \sum_{k=-\infty}^{1} \zeta(u-k)p_{0+}^{-1} + p_{1+} \zeta(u) \right)
+ \frac{p_{01}p_{0+}}{p_{0+}} \sum_{k=1}^{u} \zeta(u-k)p_{0+}^{-1} + (1 - p_{00})o(u). \quad (38)
\]

Multiplying (38) by \( s^u \) and summing from 0 to \( \infty \), one finds

\[
(1 - p_{00})\tilde{\zeta}(s) = \frac{p_{01}p_{0+}}{p_{0+}} \sum_{u=0}^{\infty} \sum_{k=-\infty}^{1} \zeta(u-k)p_{0+}^{-1} + p_{1+}\tilde{\zeta}(s)
+ \frac{p_{01}p_{0+}}{p_{0+}} \sum_{u=0}^{\infty} \sum_{k=-\infty}^{1} \zeta(u-k)p_{0+}^{-1} + (1 - p_{00})\tilde{\omega}(s)
= \frac{p_{01}p_{0+}}{p_{0+}} \sum_{u=0}^{\infty} \sum_{k=-\infty}^{1} \zeta(k)p_{0+}^{-1} + p_{1+}\tilde{\zeta}(s)
+ \frac{p_{01}p_{0+}}{p_{0+}} \sum_{u=0}^{\infty} \sum_{k=-\infty}^{1} \zeta(k)p_{0+}^{-1}+ (1 - p_{00})\tilde{\omega}(s). \quad (39)
\]

Interchanging the order of summation in (39), one finds

\[
(1 - p_{00})\tilde{\zeta}(s) = \frac{p_{01}p_{0+}}{p_{0+}} \sum_{k=0}^{\infty} \sum_{u=0}^{\infty} \zeta(k)p_{0+}^{-1} + p_{1+}\tilde{\zeta}(s)
+ \frac{p_{01}p_{0+}}{p_{0+}} \sum_{u=0}^{\infty} \sum_{k=0}^{\infty} \zeta(k)p_{0+}^{-1} + (1 - p_{00})\tilde{\omega}(s)
= \frac{p_{01}p_{0+}}{p_{0+}} \sum_{k=0}^{\infty} \sum_{u=0}^{\infty} \zeta(k)p_{0+}^{-1} + p_{1+}\tilde{\zeta}(s)
+ \frac{p_{01}p_{0+}}{p_{0+}} \sum_{u=0}^{\infty} \sum_{k=0}^{\infty} \zeta(k)p_{0+}^{-1}+ (1 - p_{00})\tilde{\omega}(s).
\]

which implies the equation in Box I.

Given that 1 is a root of the denominator, it is also a root of the numerator. It follows that

\[
-p_{10}p_{0+}(1 - p_{00})\tilde{\zeta}(p_{0+}) + (1 - p_{00})\tilde{\omega}(1)(1 - p_{00})(1 - p_{0+}) = 0,
\]

from which we conclude

\[
-p_{10}p_{0+}(1 - p_{00})\tilde{\zeta}(p_{0+}) = -(1 - p_{00})\tilde{\omega}(1)(1 - p_{00})(1 - p_{0+}). \quad (40)
\]

Substituting (40) in the equation in Box I, we find the equation in Box II.

Define

\[
h(s) = (1 - p_{00} - p_{11})(1 - sp_{0+})(s - p_{0+})
- \frac{p_{01}p_{0+}}{p_{0+}}sp_{0+}(s - p_{0+}) - p_{10}p_{1+}(1 - sp_{0+})
\]

where

\[
h(0) = (1 - p_{00} - p_{11})(p_{0+}) = 0.
\]

We know that \( h(1) = h(\rho) = 0. \) Thus, we can write

\[
h(s) = \frac{h(0)}{\rho} (s - 1)(s - \rho). \quad (41)
\]

Substituting the equation in Box II in (41), we obtain the equation in Box III.

Inverting the equation in Box III yields the equation in Box IV. From (34)-(37), we develop the following expressions for \( \tilde{\omega}(s) \) when \( w(x,y) = 1, \quad w(x,y) = x, \quad w(x,y) = y \) and \( w(x,y) = x + y \):

- **\( w(x,y) = 1 \):**
  \[
  \tilde{\omega}(s) = \frac{p_{01}}{(1 - p_{00})(1 - sp_{0+})};
  \]
\( \hat{\varphi}(s) = \frac{-p_{10}p_{11}^{(p_{0}+s)} + (1 - p_{00})\varphi(s)}{1 - p_{00} - (p_{11} + \frac{p_{01}p_{11} - p_{00}}{1 - p_{00}} + \frac{p_{01}p_{10}}{1 - p_{00}})} \)

\( \hat{\varphi}(s) = \frac{-p_{10}p_{10}^{(p_{0}+s)} + (1 - p_{00})\varphi(s)}{1 - p_{00} - (p_{11} + \frac{p_{01}p_{11} - p_{00}}{1 - p_{00}} + \frac{p_{01}p_{10}}{1 - p_{00}})} \)

\( \varphi(s) = \frac{-p_{10}p_{11}^{(p_{0}+s)} + (1 - p_{00})\varphi(s)}{1 - p_{00} - (p_{11} + \frac{p_{01}p_{11} - p_{00}}{1 - p_{00}} + \frac{p_{01}p_{10}}{1 - p_{00}})} \)

Box IV.

\( \hat{\varphi}(s) = \frac{(-1 - p_{00})\varphi(s)(1 - p_{01})(1 - p_{00} + s) + (1 - p_{00})\varphi(s)(s - p_{01})(1 - p_{00} + s)}{(1 - p_{00} - p_{11})(1 - p_{00} + s)(s - p_{01}) - \frac{p_{01}p_{10}}{1 - p_{00}}(s - p_{01}) - p_{10}p_{11}(1 - s_{01})} \)

Box II.

\( \hat{\varphi}(s) = \frac{(1 - p_{01})(1 - p_{00})(1 - s_{01})(s - p_{01}) + (1 - p_{00})\varphi(s)(s - p_{01})(1 - p_{00} + s)}{(1 - p_{00} - p_{11})(1 - p_{00} + s)(s - p_{01}) - \frac{p_{01}p_{10}}{1 - p_{00}}(s - p_{01}) - p_{10}p_{11}(1 - s_{01})} \)

Box III.

\( \varphi(s) = \frac{(-1 - p_{00})\varphi(s)(1 - p_{01})(1 - p_{00} + s) + (1 - p_{00})\varphi(s)(s - p_{01})(1 - p_{00} + s)}{(1 - p_{00} - p_{11})(1 - p_{00} + s)(s - p_{01}) - \frac{p_{01}p_{10}}{1 - p_{00}}(s - p_{01}) - p_{10}p_{11}(1 - s_{01})} \)

Box I.

\( \psi(u) = \frac{\mu(u) - \frac{p_{01}p_{10}}{1 - p_{00}} + p_{00}(\rho - p_{10})}{(1 - p_{01})(1 - p_{00})(1 - s_{01})(s - p_{01})} \left( \frac{1}{\rho} \right)^{u+1} \)

where

\( \psi(0) = \frac{p_{01}(\rho - p_{10}) - p_{11}(1 - p_{00} + \rho)}{(1 - p_{00} - p_{11})(1 - p_{00} + \rho)} \)

Box IV.

\( w(x, y) = x: \)

\( \varphi(s) = \frac{p_{01}p_{10}^{s}}{(1 - p_{00})(1 - s_{01})(1 - p_{01})} \)

\( w(x, y) = y: \)

\( \varphi(s) = \frac{p_{01}p_{10}^{s}}{(1 - p_{00})(1 - s_{01})(1 - p_{01})} \)

\( w(x, y) = x + y: \)

\( \omega(u) = \frac{p_{01}p_{10}^{s}}{(1 - p_{00})(1 - s_{01})(1 - p_{01})} + \frac{p_{01}}{(1 - p_{00})(1 - s_{01})(1 - p_{01})} \)

\( \varphi(s) = \frac{p_{01}p_{10}^{s}}{(1 - p_{00})(1 - s_{01})(1 - p_{01})} \times \left( \frac{p_{01}p_{10}^{s}}{(1 - p_{00})(1 - s_{01})(1 - p_{01})} + \frac{1}{1 - p_{00} + p_{01} + \rho} \right) \)

\( \psi(u) = \frac{(-1 - p_{00})(1 - p_{01})(1 - p_{00} + s) + (1 - p_{00})\varphi(s)(s - p_{01})(1 - p_{00} + s)}{(1 - p_{00} - p_{11})(1 - p_{00} + s)(s - p_{01}) - \frac{p_{01}p_{10}}{1 - p_{00}}(s - p_{01}) - p_{10}p_{11}(1 - s_{01})} \left( \frac{1}{\rho} \right)^{u+1} \)

where

\( \psi(0) = \frac{p_{01}(\rho - p_{10}) - p_{11}(1 - p_{00} + \rho)}{(1 - p_{00} - p_{11})(1 - p_{00} + \rho)} \)

Example 4. We assume that \((X, W)\) has a bivariate geometric distribution of type 2. The r.v. \(X\) has a marginal geometric distribution with parameter \(q_{1} = p_{11} = \frac{1}{2}\) \((q_{2} = p_{10} = \frac{1}{2}\) such that \(E[X] = \frac{1}{q_{1}} = 1.5\) \(E[W] = \frac{1}{q_{2}} = 1.8\). The premium rate \(c\) is equal to 1, including a security margin of \(\eta = 20\%.\) The bivariate Bernoulli distribution of \((I, F)\) is defined with a Frank copula (dependence parameter \(= \alpha\). It implies that

\( p_{00} = \frac{-1}{\alpha} \ln \left( 1 + \frac{e^{-\alpha p_{01}} - 1}{e^{-\alpha p_{01}} - 1} \right) \)

and we find the other probabilities with \(p_{01} = p_{02} - p_{00}, p_{10} = p_{01} + p_{00}, p_{11} = 1 - p_{00} - p_{01} - p_{10}\). In Table 3, we give the values of \(\varphi(0)\) and \(\psi(0)\), where \(\alpha(0)\) and \(\psi(0)\) stand for the values of \(\alpha\) and \(\psi(0)\) with a dependence parameter equal to \(\alpha\). When \(\alpha = 0\), we are in the context of independence and the resulting model is the compound binomial risk model. The explicit expression for the ruin probability \(\psi(0) = \frac{\varphi(1-q_{2})}{\varphi(1-q_{2})^{2}}\) is given in Willmot (1993). We provide in the table below the values of \(\alpha\) and \(\psi(0)\) for \(\alpha = -10, -5, 0, 5\) and 10:
The resulting ruin probabilities are given in Table 4. In Fig. 3, we depict the values of $\psi^{(u)}(\alpha)$ for $u = 0, 1, \ldots, 30$. For any fixed value of initial surplus $u$, the ruin probabilities increase as the dependence parameter $\alpha$ decreases.

5.2. Finite support

In this section, we assume that $W \in \{1, 2, \ldots, m + 1\}, X \in \{1, 2, \ldots, l + 1\}$, with $l, m \in \mathbb{N}^+$. The p.m.f. of the r.v. $Y$ is computed with (17) for $y \in \{-m, -m + 1, \ldots, 0, 1, \ldots, l - 1, l\}$. Its p.g.f. is given by $f_Y(s) = \sum_{k=-m}^{l} f_Y(k) s^k$.

Letting $s = \frac{1}{t}$, the Lundberg equation in (19) becomes

$$\sum_{k=-m}^{l} f_Y(k) \left(\frac{1}{t}\right)^k = 1$$

which we multiply by $t^i$ to obtain

$$\sum_{k=-m}^{l} f_Y(j-m) t^{m+i-j} = t^i.$$  (42)

We assume that there are no multiple roots with absolute value less than 1 to (42). Then, (42) has $l+m$ roots (counting multiplicity), denoted $r_1, r_2, \ldots, r_1, r_{l+1}, \ldots, r_{l+m}$. From Lemma 2.2 of Liu and Guo (2006), we have

$$|r_1| \leq |r_2| \leq \cdots \leq |r_l| < 1 \leq |r_{l+1}| \leq \cdots \leq |r_{l+m}|.$$

Now, for $u \in \{0, 1, 2, \ldots, l - 1\}$, we can see that

$$\zeta(u) = \sum_{k=-m}^{l} \zeta(u-k) f_Y(k) + o(u),$$

where

$$o(u) = \sum_{k=l+1}^{l+m} \sum_{i=1}^{m+1} f_Y(t+k, t) w(u + t - 1, k - u).$$

For $u \in \mathbb{N} \setminus \{0, 1, 2, \ldots, l - 1\}$, $o(u) = 0$ and

$$\zeta(u) = \sum_{k=-m}^{l} \zeta(u-k) f_Y(k).$$  (43)

We want to establish the same relation for $\zeta(0), \zeta(1), \ldots, \zeta(l-1)$ as in (43). For that purpose, we define $\zeta(-1), \zeta(-2), \ldots, \zeta(-l)$ as the solutions to the following system of equations:

$$f_Y(l) \zeta(-1) = w(0)$$

$$f_Y(l-1) \zeta(-1) + f_Y(l) \zeta(-2) = w(1)$$

$$\ldots$$

$$\sum_{j=1}^{l} f_Y(k) \zeta(-k) = w(l-1).$$

When $w(x, y) = 1$, we have that $w(u) = 1$ and $\zeta(u) = 1$ for $u \in \{0, 1, 2, \ldots, l - 1\}$. Having defined $\zeta(-1), \zeta(-2), \ldots, \zeta(-l)$, $\zeta(u)$ fulfills (43) for $u \in \mathbb{N}$.

The characteristic equation associated to (43) is given by (42). Using the results of Liu and Guo (2006) and letting $\rho_i = 1/r_i$ ($i = 1, 2, \ldots, l$), the explicit expression for $\zeta(u)$ ($u \in \mathbb{N}$) is given by

$$\zeta(u) = \sum_{i=1}^{l} c_i \left(\frac{1}{\rho_i}\right)^u.$$  (44)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$-10$</th>
<th>$-5$</th>
<th>$0$ (independence)</th>
<th>$5$</th>
<th>$10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>1.2348944</td>
<td>1.2506844</td>
<td>1.333333333</td>
<td>1.612486</td>
<td>2.046872</td>
</tr>
<tr>
<td>$\psi(0)$</td>
<td>0.7416877</td>
<td>0.6993433</td>
<td>0.625</td>
<td>0.430269</td>
<td>0.2328255</td>
</tr>
</tbody>
</table>

The coefficients $c_1, c_2, \ldots, c_l$ in (44) are uniquely determined by the following system of equations:

$$c_1 \rho_1^l + c_2 \rho_2^l + \cdots + c_l \rho_l^l = \zeta(-1)$$

$$c_1 \rho_1^{l-1} + c_2 \rho_2^{l-1} + \cdots + c_l \rho_l^{l-1} = \zeta(-l + 1)$$

$$\ldots$$

$$c_1 \rho_1 + c_2 \rho_2 + \cdots + c_l \rho_l = \zeta(-l).$$

When $w(x, y) = 1$ and using Corollary 6.4.5 of Rolski et al. (1999), the $c_i$’s are explicitly given by

$$c_i = \frac{1}{(\rho_i - 1) \prod_{j=1}^{l} \left(1 - \frac{1}{\rho_j}\right) \prod_{j \neq i} \rho_j}.$$  (45)

for $i = 1, 2, \ldots, l$.

Note that a recursive equation can be developed for $\zeta(u)$. From (44), we have

$$\hat{\zeta}(s) = \sum_{u=0}^{\infty} \zeta(u) s^u = \sum_{u=0}^{\infty} \sum_{i=1}^{l} c_i \left(\frac{1}{\rho_i}\right)^u = \sum_{i=1}^{l} c_i \frac{1}{1 - s/\rho_i}.$$  (46)

$$\hat{f}_l(s) = \frac{1}{1 - \frac{s}{\rho_i}}$$  (47)
is the p.g.f. associated with the mass probability function $f_i$ of a r.v. $L$. The support of $L$ is $\{1, 2, \ldots, l\}$. Given (46), $\hat{\zeta}(s)$ satisfies the following relation

$$\hat{\zeta}(s) = \hat{\zeta}(s) \hat{f}_L(s) + \hat{H}(s),$$

which leads to

$$\hat{\zeta}(u) = \sum_{j=1}^{\min(u,l)} \phi(u-j) f_L(j) + H(u),$$

for $u \in \mathbb{N}^+$ and $\zeta(0) = H(0) = \sum_{i=1}^{l} \epsilon_i$. Inverting (47) leads to

$$f_L(1) = \frac{1}{\rho_1},$$

$$f_L(2) = -\frac{1}{\rho_1} \sum_{i=1}^{l-1} \frac{1}{\rho_i} f_L(i) + \frac{1}{\rho_1},$$

$$\vdots$$

$$f_L(l) = (-1)^{l-1} \prod_{i=1}^{l} \frac{1}{\rho_i}.$$

As explained in e.g. Rolski et al. (1999), $\psi(u)$ is equal to $\Pr(L^* > u)$ where $L^* = \max(0, V_1, V_2, \ldots)$. It is proven in Theorem 6.3.3 of Rolski et al. (1999) that $L^*$ has a compound geometric distribution i.e. $L^* = \sum_{k=1}^{N} L_k$, where $L_k$ is the $k$th ascending ladder height of the random walk $V$ defined in (18) and $L_k$ has the same distribution as $L$. This allows us to write $1 - \psi(u) = \sum_{k=1}^{\infty} (1 - \gamma)^k f_{L_k}(u)$ ($u \in \mathbb{N}$) where

$$\gamma = 1 - \psi(0) = 1 - \sum_{i=1}^{l} \epsilon_i,$$

$$= 1 - \prod_{i=1}^{l} \left(\frac{1}{\rho_i} - 1\right) \prod_{j=1, j \neq i}^{l} \frac{1}{\rho_j}.$$

from (45). See also Willmot and Lin (2001) for results on the behavior of compound geometric distributions.

**Example 5.** We suppose that $(X, W)$ has a bivariate geometric distribution which is defined with a Frank copula (dependence parameter $= \alpha$). The r.v. $X$ ($W$) has a marginal geometric distribution with parameter $q_1 = \frac{1}{15}$, $q_2 = \frac{1}{18}$ such that $E[X] = \frac{1}{q_1} = 1.5$ and $E[W] = \frac{1}{q_2} = 1.8$. The premium rate $c$ is equal to 1, including a security margin of $\eta = 20\%$. In order to apply the results of this sub-section for the computation of the ruin probabilities, we calculate $f_{X,W}(x, t)$ with (10) for $x = 1, 2, \ldots, 25$ and $t = 1, 2, \ldots, 25$. It implies that $l = m = 24$. For the chosen values of $q_1$ and $q_2$, we verify that $\sum_{x=1}^{25} \sum_{t=1}^{25} f_{X,W}(x, t) = 1$ for all considered values of $\alpha$. The values of the ruin probabilities for $\alpha = -10, -5, 0, 5, 10$ are provided in Table 5 and depicted in Fig. 4.

**Example 6.** We consider a bivariate distribution for $(X, W)$ defined with a Frank copula and with the same marginals as the ones used in Example 1 of Li (2005a). The claim amount r.v. $X$ has a mixture of two geometric distributions with

$$f_X(x) = 0.6 \times \left(\frac{1}{3}\right)^{x-1} + 0.4 \times \left(\frac{1}{3}\right)^{x-1},$$

for $x \in \mathbb{N}^+$ and the claim waiting time r.v. $W$ has a shifted negative binomial distribution with

$$f_W(x) = f \left(\frac{2}{3}\right)^{x} \left(\frac{1}{3}\right)^{x-1},$$

for $t \in \mathbb{N}^+$. It follows that $E[X] = 1.8$ and $E[W] = 1.2$. Since the premium rate $c = 1$, the security margin $\eta$ is equal to 10\%. As in Example 5, we calculate $f_{X,W}(x, t)$ with (10) for $x = 1, 2, \ldots, 25$ and $t = 1, 2, \ldots, 25$. It implies that $l = m = 24$. We verify that $\sum_{x=1}^{25} \sum_{t=1}^{25} f_{X,W}(x, t) = 1$ for all considered values of $\alpha$. The values of the ruin probabilities for $\alpha = -10, -5, 0$ (independence), 5 and 10 are provided in Table 6 and depicted in Fig. 5.

□
5.3 Comments

We conclude this section with two comments. First, we observe in the numerical Examples 4–6 that

$$\psi^{(10)}(u) \leq \psi^{(5)}(u) \leq \psi^{(0)}(u) \leq \psi^{(-5)}(u) \leq \psi^{(-10)}(u),$$

for all $u \in \mathbb{N}$. The impact of the dependence relation between the r.v.’s $X$ and $W$ on the ruin probabilities can be interpreted as follows. In the context of positive (negative) dependence, as the time elapsed since the last claim increase (decreases), the probability of having an important claim increases. It implies that the probability that the insurance company has enough premium income to pay the claim is higher (lower) and the ruin probability is lower when a positive (negative) dependence relation is assumed. The impact on the ruin probabilities is more significant when the positive (negative) relation becomes stronger.

As a second comment, we want to mention that it is possible to find an explicit expression for $\psi(u)$ when $(X, W)$ has a bivariate geometric distribution which is defined with the countermonotonic copula as in Section 4.2.5. The r.v.’s $X$ and $W$ have marginal geometric distribution with parameter $q_1$ and $q_2$ respectively and the parameters $q_1$ and $q_2$ are fixed such that $1 - (1 - q_1) - (1 - q_2) = q_1 + q_2 - 1 > 0$. Note that the form of $f_Y$ in (28) is similar (except for some coefficients) to the one of the $f_Y$ in (23) for the bivariate geometric distribution of type 2. Then, we give below the expression of $\psi(u)$ without providing too many details of the derivation since its steps are similar to the ones used in Section 5.1. For $u = 0, 1, 2, \ldots$, we have

$$\psi(u) = \sum_{k=0}^{\infty} f_X(k+1) \psi(k+1),$$

due to condition (27). When $w(x, y) = 1$, we have $\omega(u) = 1 - F_Y(u) = p_1$. Then, (48) becomes

$$\psi(u) = \sum_{k=0}^{\infty} f_X(k+1) \psi(k+1).$$
Fig. 4. Values of $\psi^{(\alpha)}(u)$ for the bivariate geometric distribution defined with the Frank copula for $\alpha = -10, -5, 0, 5,$ and $10$.

Fig. 5. Values of $\psi^{(\alpha)}(u)$ for the bivariate distribution with Frank copula (for $\alpha = -10, -5, 0, 5,$ and $10$) and two specific marginals (shifted negative binomial and mixture of two geometrics).

6. Approximation of the continuous-time renewal risk model

An interesting advantage of the discrete-time renewal risk model with dependence is that it can be used to approximate the corresponding continuous-time renewal risk model with dependence. We briefly describe the continuous-time renewal risk model with dependence and then present the approximation.

6.1. Continuous time renewal risk model

In the continuous-time renewal risk model with an arbitrary dependence structure between the interclaim time and the subsequent claim size, we assume a sequence of i.i.d. bivariate random vectors $(W^c_j, X^c_j)$ for $j \in \mathbb{N}^+$. As in Albrecher and Teugels (2006), we assume that the dependence structure between the r.v.'s $W^c_j$ and $X^c_j$ is expressed through a copula $C$. The claim number
process $N^t = \{N^t_i, t \geq 0\}$ is a renewal process. The total claim amount process $S^t = \{S^t_i, t \geq 0\}$ is defined as $S_t = \sum_{i=1}^{N_t} X_i^t$.

Finally, we define the surplus process $U^t = \{U^t_i, t \geq 0\}$ as $U^t_i = u + ct - S^t_i$ where $u$ ($u \in \mathbb{N}$) is the initial surplus level and $c$ ($c \geq 0$) is the level premium rate. We assume that the premium rate is equal to 1. Let $\tau = \inf_{t \geq 0}(U^t_i < 0)$ be the time of ruin with $\tau = \infty$ if $U^t_i \geq 0$ for all $t \geq 0$ (i.e. ruin does not occur). The corresponding infinite time ruin probability is denoted by $\psi(u)$. In the continuous-time renewal risk model, it is clear that the increments $\{X_i^t - cW_i^t \}, i \in \mathbb{N}^t$ of the surplus process are still independent. With a martingale argument, we obtain an exponential upper bound for the ruin probability $\psi(u) \leq e^{-\psi u}$, where $\psi > 0$ if it exists. This is the strictly positive solution to $E[e^{\psi(X^t - cW^t)}] = 1$. Depending on the choice of copula and marginals, it might be difficult to have an explicit expression for $\psi(u)$ and $E[e^{\psi(X^t - cW^t)}]$.

6.2. Approximation

Our objective is to propose a procedure which relies on the infinite-time ruin probabilities and the adjustment coefficient in the discrete time renewal risk model with dependence to derive both an upper and a lower bound for $\{\psi(u), u \in \mathbb{N}\}$ and the adjustment coefficient within the continuous time renewal risk model with dependence. This procedure is explained in detail in Cossette et al. (2006).

We assume that $[x]$ represents the largest integer $\leq x$, $Y \overset{d}{=} Z$ indicates that $Y$ is equal in distribution to $Z$ and $Y \overset{\leq}{=} Z$ if the r.v. $Y$ dominates the r.v. $Z$ under the stochastic dominance order (see e.g. Shaked and Shanthikumar (1994) and Müller and Stoyan (2002)). We assume that the premium rate $c = 1$ in the continuous time renewal risk model with dependence.

We consider two specific (time-modified) discrete time renewal risk models. In the first one, the surplus process is defined as $U_0^t = mu$ and

$$U_k^t = mu - \sum_{i=1}^{k} (X_i^t - W_i^t), \quad \text{for} \quad k, m \in \mathbb{N}^t, \tag{49}$$

where $X_i^t \overset{d}{=} |mX_i^c| + 1$, $W_i^t \overset{d}{=} |mW_i^c| + 2$ and $c = 1$. In the second one, the surplus process is defined as $U_0^t = mu$ and

$$U_k^t = mu - \sum_{i=1}^{k} (X_i^t - W_i^t), \quad k \in \mathbb{N}^t \tag{50}$$

for $m \in \mathbb{N}^t$ where $X_i^t \overset{d}{=} |mX_i^c| + 2$, $W_i^t \overset{d}{=} |mW_i^c| + 1$ and $c = 1$. The distributions of the random vectors $(X_i^*, W_i^*)$, $(X_i^c, W_i^c)$, and $(X_i, W_i)$ are defined with the same copula.

Both discrete time renewal risk models are derived from the continuous-one by discretizing the distributions of $X$ and $W$. Finally, we denote by $\psi^*$ (and $\rho^*$) and $\psi^c$ (and $\rho^c$) the infinite-time ruin probability (and the adjustment coefficient) associated respectively to the surplus processes $U^*$ and $U^c$.

To compare adjustment coefficients in the continuous time and the discrete-time renewal risk models with dependence, let $\rho^c$ be equal to $e^{\psi^c}$. Since $R^c > 0$, we have $\rho^c > 1$. The following proposition is an extension of Proposition 8 in Cossette et al. (2006).

**Proposition 7.** In the continuous time renewal risk model with dependence with $c^* = 1$, the adjustment coefficient and the infinite-time ruin probability satisfy

$$\rho^* \geq \rho^c \geq \rho^1 \tag{51}$$

and

$$\psi^*(mu) \leq \psi^c(u) \leq \psi^1(mu), \tag{51}$$

for $u \in \mathbb{N}$ and $\forall m \in \mathbb{N}^+$. 

**Proof.** The bounds on $\psi^c$ and $\rho^c$ are obtained under the stochastic dominance. We have $X_i^c \overset{\leq}{=} X_i^c \overset{\leq}{=} X_i^c \overset{\leq}{=} -W_i^c \overset{\leq}{=} -W_i^c \overset{\leq}{=} -W_i^c$. Let $C(u, v)$ be the same copula associated to the r.v.’s $(X_i^c, W_i^c)$, $(X_i, W_i)$, and $(X_i, W_i)$. By Theorem 2.4.4 (item 1) of Nelsen (2006), the common copula associated to $(X_i^c, W_i^c)$, $(X_i, W_i)$ and $(X_i, W_i)$ is $u = C(u, 1 - v)$. Then, from Theorem 3.3.8 of Müller and Stoyan (2002) we deduce

$$X_i^c - W_i^c \overset{\leq}{=} m(X_i^c - W_i^c) \overset{\leq}{=} X_i^c - W_i^c \tag{52}$$

Applying Theorem 1.2.17 of Müller and Stoyan (2002) with (52), we conclude that

$$\max_{k \in \{1, \ldots, m\}} \left\{ \sum_{i=1}^{k} (X_i^c - W_i^c) \right\} \leq \max_{k \in \{1, \ldots, m\}} \left\{ \sum_{i=1}^{k} m(X_i^c - W_i^c) \right\} \leq \max_{k \in \{1, \ldots, m\}} \left\{ \sum_{i=1}^{k} (X_i^c - W_i^c) \right\} \leq \max_{k \in \{1, \ldots, m\}} \left\{ \sum_{i=1}^{k} (X_i^c - W_i^c) \right\},$$

which implies

$$\psi_1^c(mu) \leq \psi_c^c(u) \leq \psi_1^c(mu), \tag{53}$$

for $l \in \mathbb{N}^+$ where $\psi_1^c$, $\psi_c^c$ and $\psi_1^c$ are the ruin probabilities on or before the $l$th claim in their respective risk models. Since (53) holds for all $l$, we derive (51) by letting $l \to \infty$ in (53).

From (52), we have $E[s^{X_i^c - W_i^c}] \leq E[s^{X_i - W_i}] \leq E[s^{X_i^c - W_i^c}]$, which implies $\rho^c \geq \rho^* \geq \rho^1$. □

The value of $m (m \in \mathbb{N}^+)$ must be chosen large enough to ensure that the relative security margin $\rho$ remains positive after the discretization. Note that the larger is $m$ in Proposition 7, the tighter will be the upper and lower bounds for the infinite-time ruin probabilities.

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