Ruin Theory in risk models based on time series for count random variables

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A celebration of Hans Gerber’s contributions (Waterloo, Canada)

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1. Introduction

- The objective of the talk is examine an extension of the classical discrete-time risk model
- Our extension is loosely inspired on Gerber’s own extension of this model
- Gerber’s extension is based on gaussian linear time series
- Our extension is based on time series for count data
- We examine the computation of the following ruin measures:
  - Adjustment coefficient
  - Gerber-Shiu fonction on finite-time horizon
  - Joint probabilities of time of ruin, deficit at ruin and surplus before ruin
  - Dynamic VaR and TVaR
- In this talk, we "celebrate" among numerous Gerber’s contributions
  - Gerber’s gaussian linear risk model
  - So-called Gerber-Shiu function
  - Computation of various ruin measures
The classical discrete time risk model is due to De Finetti (1957) (see also e.g. Bühlmann (1970), Gerber (1979) and Dickson (2005)).

We consider an insurance portfolio

Aggregate claim amount in period $k \in \mathbb{N}^+$: r.v. $W_k$

$\mathcal{W} = \{W_k, k \in \mathbb{N}^+\}$ = sequence of i.i.d. r.v.’s

Premium income per period: $\pi = (1 + \eta) E[W]$

Strictly positive security margin: $\eta > 0$
2. Classic discrete-time risk model

- Surplus process: $U = \{ U_k, k \in \mathbb{N} \}$
  - for $k = 0$, $U_0 = u = \text{initial surplus}$
  - for $k \in N^+$, $U_k = U_{k-1} + \pi - W_k = u - \sum_{j=1}^{k} (W_j - \pi)$

- Time of ruin: r.v. $\tau$ with
  $$\tau = \begin{cases} 
  \inf_{k \in \{1,2,3,\ldots\}} \{ k, U_k < 0 \}, & \text{if } U_k \text{ goes below } 0 \text{ at least once} \\
  \infty, & \text{if } U_k \text{ never goes below } 0
  \end{cases}$$

- Finite-time ruin probability over $n$ periods: $\psi(u, n) = \Pr(\tau \leq n)$
- Infinite-time ruin probability: $\psi(u) = \lim_{n \to \infty} \psi(u, n)$
2. Classic discrete-time risk model

- Net loss in period $j \in \mathbb{N}^+$: $L_j = (W_j - \pi)$
- $L_1, L_2, \ldots$ i.i.d. with $E[L_j] = E[L_j] - \pi < 0$ (since $\eta > 0$)
- Random walk with negative drift: $Y = \{Y_k, k \in \mathbb{N}\}$: $Y_0 = 0$ and
  \[ Y_k = \sum_{j=1}^{k} L_j \]

- Maximum net cumulative loss process: $Z = \{Z_k, k \in \mathbb{N}\}$
  \[ Z_k = \max_{j=0,1,2,\ldots,k} \{Y_j\} \]

- Finite-time ruin probability over $n$ periods:
  \[ \psi(u, n) = \Pr(\tau \leq n) = \Pr(Z_n > u) = 1 - F_{Z_n}(u) \]
2. Classic discrete-time risk model

- The distribution of $Z_n$ has a mass at 0
- Risk management: the knowledge of $F_{Z_n}$ can be used to determine the initial capital
- We consider 2 risk measures based on $Z_n$
  - Dynamic VaR
  - Dynamic TVaR
- Dynamic VaR:
  $$\text{VaR}_\kappa (Z_n) = F_{Z_n}^{-1} (\kappa)$$
  (see e.g. Albrecher (2009), Truffin et al. (2010), Cossette et al. (2013)).
- Interpretation: $\text{VaR}_\kappa (Z_n) =$ capital required such that the probability that the maximum $Z_n$ of the random walk $Y$ over the first $n$ periods exceeds that capital is equal to $1 - \kappa$. 
Dynamic TVaR:

\[
TVaR_\kappa (Z_n) = \frac{1}{1 - \kappa} \int_{1 - \kappa}^{\infty} \text{VaR}_\nu^{(d)} (Z_n) \, dv
\]

equals to

\[
E \left[ Z_n \times 1\{Z_n > \text{VaR}_\kappa (Z_n)\} \right] + \text{VaR}_u (Z_n) (F_{Z_n} (\text{VaR}_\kappa (Z_n)) - \kappa) \frac{1}{1 - \kappa}
\]

(Cossette et al. (2013))

Compared to the dynamic VaR, the dynamic TVaR has the advantage of being more sensitive to the stochastic behavior of \( Z_n \) in the tail of its distribution.
3. Gerber’s linear gaussian (discrete time) risk model

- Some papers consider various models with temporal dependence.
- Gerber (1982) proposes extension:
  - Title: Ruin Theory in the linear model, IME 1982
  - The risk model is based on gaussian linear time series (e.g. AR, ARMA, etc.)
  - Gerber examines the approximation of ruin probabilities

- Variants and extension
  - Promislow (1991): upper bounds for $\psi(u)$ in a similar risk model
  - Christ and Steinebach (1995): empirical mgf type estimator of the adjustment coefficient in Gerber’s risk model.
  - Yang and Zhang (2003): both exponential and non-exponential upper bounds for $\psi(u)$ in an extension to Gerber’s risk model.
As stated in almost all actuarial textbooks, compound distributions are the corner stones of several risk models in risk theory.

The linear risk models such as the Gaussian AR(1) and ARMA\((p, q)\) do not lead to compound distributions for the rvs \(W\).

We adapt the idea of Gerber.

We assume that \(W_k\) follows a compound distribution

\[
W_k = \begin{cases} 
\sum_{j=1}^{N_k} B_{k,j}, & N_k > 0 \\
0, & N_k = 0
\end{cases}
\]

- r.v. \(N_k\): number of claims in period \(k\)
- \(\{B_{k,j}, j \in \mathbb{N}^+\}\) = individual claim amounts in period \(k\) (positive r.v.'s)
- \(\{B_{k,j}, j \in \mathbb{N}^+\}\) = sequence of i.i.d. r.v's and also independent of \(N_k\)

Premium income per period: \(\pi = (1 + \eta) \mathbb{E}[W]\)

Strictly positive security margin: \(\eta > 0\)
We consider risk models based on compound distributions assuming time series models for count data for \( \mathbf{N} = \{N_k, k \in \mathbb{N}^+\} \).

4 categories of time series models for count data for \( \mathbf{N} \).

Models based on thinning:

- \( \mathbf{N} = \text{Poisson INAR}, \text{Poisson INMA}, \text{Poisson INARMA}, \) etc.
- These models are based on appropriate thinning operations which replace the scalar multiplications by a fraction in the Gaussian ARMA time series
  (see e.g. Mckenzie (1986, 1988, 2003), and Joe (1997))
- Freeland (1998) and Freeland and McCabe (2004) : Worker’s Compensation Board of British Columbia
- Kremer (1995) adapts the theory of INAR processes to the context of IBNR-predictions
4. Ruin theory based on count ...

- Other types of models of times series for count data:
  - Models based on Markov chains. $N$ = Markov chain of order 1 or more.
  - Models based on a specific conditional distribution with stochastic parameters.
    - $N$ has a dependence structure based on an underlying process $\Theta$
    - $\Theta$ = ARMA time series or hidden discrete time Markov chain defined on a finite time space
    - Malyshkina, Mannering and Tarko (2009) explore two-state Markov switching count data models to study accident frequencies
  - Models based on copulas: marginals are fixed and the dependence structure of $N$ is based on a copula (see e.g. Frees and Wang (2006))

- Reviews on time series models for count data:
  - Cameron and Trivedi (1998)
  - Kedem and Fokianos (2002)

- In this talk, we focus on the case $N$ = Poisson INAR1 process
Dynamic of $N = \{N_k, k \in \mathbb{N}^+\}$

$$N_k = \alpha \circ N_{k-1} + \varepsilon_k,$$ \hspace{1cm} (1)

for $k = 2, 3, \ldots$

- $\varepsilon = \{\varepsilon_k, k \in \mathbb{N}^+\}$ : sequence of i.i.d. r.v.'s with $\varepsilon_k \sim Pois ((1 - \alpha) \lambda)$ where $\alpha \in [0, 1]$.

- $\alpha \circ N_{k-1}$ :
  - The operator "$\circ$" is used in models based on thinning
  - We have
  $$\alpha \circ N_{k-1} = \sum_{i=1}^{N_{k-1}} \delta_{k-1,i},$$

  where $\{\delta_{k-1,i}, i = 1, 2, \ldots\}$ is a sequence of i.i.d Bernoulli r.v's with mean $\alpha$ and independent of $N_{k-1}$

- $\alpha$ = dependence parameter

Given (1), $N$ is stationary with $N_k \sim Pois (\lambda)$. 
Risk Model – Poisson INAR(1)

- Interpretation in an insurance context:
  \[ N_k = \text{nb of claims from period } k - 1 \text{ leading to claims in period } k + \text{nb of new claims in period } k \]
  \[ = \alpha \circ N_{k-1} + \varepsilon_k \]

- Autocorrelation function for \( N \):
  \[ \gamma_N (h) = \alpha^h, \; h \geq 1, \]
  with \( \gamma_N (1) \in [0, 1) \).

- Covariance between \( W_k \) and \( W_{k+h} \):
  \[ \text{Cov} (W_k, W_{k+h}) = \lambda \alpha^h E [B]^2, \]
  for \( h \geq 1 \).
Paths of $N(\alpha = 0)$
Paths of $N(\alpha = 0.4)$
Paths of $N(\alpha = 0.99)$
Paths of $V (\alpha = 0)$
Paths of $V (\alpha = 0.4)$
Risk Model – Poisson INAR(1)

- Paths of $V$ ($\alpha = 0.99$)
Paths of $Z(\alpha = 0)$
Paths of $Z (\alpha = 0.4)$
Paths of $Z (\alpha = 0.99)$
Adjustment coefficient and asymptotic expression

- Define the convex function
  \[ c_n(r) = \frac{1}{n} \ln \left( \mathbb{E} \left[ e^{rV_n} \right] \right) = \frac{1}{n} \ln \left( \mathbb{E} \left[ e^{r(S_n-n\pi)} \right] \right) =, \tag{2} \]

  where \( S_n = \sum_{k=1}^{n} W_k \)

- Lundberg adjustment coefficient: \( \rho \) is the solution to
  \[ c(r) = \lim_{n \to \infty} c_n(r) = 0. \tag{3} \]

  (see e.g. Nyrhinien (1998), Müller and Pflug (2001))

- Asymptotic Lundberg-type result:
  \[ \lim_{u \to \infty} -\frac{\ln (\psi(u))}{u} = \rho, \]

- Adjustment coefficient \( \rho \): measure of dangerousness of an insurance portfolio.

- The expression of \( c_n(r) \) depends on the temporal dependence structure for \( W \).
Adjustment coefficient – Poisson INAR(1)

- **Poisson AR(1).** Assuming that $\alpha M_B(r) < 1$, the expression for $c(r)$ is given by

$$
c(r) = \frac{(1 - \alpha)^2 \lambda M_B(r)}{1 - (\alpha M_B(r))} - (1 - \alpha) \lambda - r \pi.
$$

- **Poisson AR(1).** Impact of dependence parameter $\alpha$.
  - If $\alpha < \alpha'$, then $\rho > \rho'$.
  - Dangerousness increases with the dependence parameter $\alpha$.

- **Poisson AR(1) + Exponential claims.** Assume that $B \sim \text{Exp}(\beta)$ with mean $\frac{1}{\beta}$. Then, we have

$$
\rho = \frac{(1 - \alpha) \beta \eta}{1 + \eta},
\quad (4)
$$

where $\alpha \in [0, 1)$. Special case: if $\alpha = 0$, $\rho = \frac{\beta \eta}{(1 + \eta)}$ (independence).

- (Cossette et al. (2010)).
**Example.**

- \( \lambda = 0.4, \ B \sim \text{Exp} \left( \beta = \frac{1}{2} \right) \) and \( \eta = 0.25 ; \ c = 1 \)
- approximation based on 5000 crude MC simulations, 1000 periods

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( E[Z_{1000}] )</th>
<th>( \text{Var}(Z_{1000}) )</th>
<th>( \text{VaR}<em>{0.95}(Z</em>{1000}) )</th>
<th>( \text{VaR}<em>{0.99}(Z</em>{1000}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>9.646</td>
<td>151.904</td>
<td>35.018</td>
<td>55.493</td>
</tr>
<tr>
<td>0.8</td>
<td>34.777</td>
<td>1964.363</td>
<td>127.714</td>
<td>191.145</td>
</tr>
</tbody>
</table>

- approximation with asymptotic expression

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \rho )</th>
<th>( -\frac{1}{\rho} \ln(0.05) )</th>
<th>( -\frac{1}{\rho} \ln(0.01) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.08</td>
<td>37.447</td>
<td>57.565</td>
</tr>
<tr>
<td>0.8</td>
<td>0.02</td>
<td>149.787</td>
<td>230.258</td>
</tr>
</tbody>
</table>
Example (suite)

- Approximation of $\psi(u)$ using importance sampling

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\rho$</th>
<th>$\tilde{\psi}(0)$</th>
<th>$\tilde{\psi}(10)$</th>
<th>$\tilde{\psi}(20)$</th>
<th>$\tilde{\psi}(30)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
<td>0.771769</td>
<td>0.280084</td>
<td>0.103552</td>
<td>0.037806</td>
</tr>
<tr>
<td>0.2</td>
<td>0.08</td>
<td>0.816092</td>
<td>0.364378</td>
<td>0.164038</td>
<td>0.072333</td>
</tr>
<tr>
<td>0.4</td>
<td>0.06</td>
<td>0.850753</td>
<td>0.463960</td>
<td>0.255592</td>
<td>0.140231</td>
</tr>
</tbody>
</table>

- $\psi(u) \uparrow$ as $\alpha \uparrow$
- Adaptation from e.g. Asmussen (2003).
We present recursive formulas for the computation of finite-time ruin measures.

Additional assumptions: $\pi \in \mathbb{N}^+, u \in \mathbb{N}$, rv $B$ is defined on $\mathbb{N}$ (to simplify the notation).

All formulas can be easily adapted to the case when the support is $\{0, 1h, 2h, \ldots\}$ with $h > 1$.

Notation:
- $\Pr (N_1 = j) = p_j, j \in \mathbb{N}$
- $\Pr (N_2 = k_2 | N_1 = k_1) = p_{k_1,k_2}, k_1, k_2 \in \mathbb{N}$

We compute $\psi(u, n)$ with

$$\psi(u, n) = \sum_{k_1=0}^{\infty} p_{k_1} \psi(u, n | N_1 = k_1).$$

where

$$\psi(u, n | N_1 = k_1) = \Pr (\tau \leq n | N_1 = k_1).$$
Finite-time ruin probability

- For $n = 1$, we have

\[ \psi(u, 1|N_1 = k_1) = \sum_{l=u+c+1}^{\infty} f(l, k_1), \]

where

\[ f(j; k_1) = \begin{cases} 
\Pr(B_1 + \ldots + B_{k_1} = j), & \text{if } k_1 > 0 \\
1, & \text{if } k_1 = 0 \text{ and } j = 0 \\
0, & \text{if } k_1 = 0 \text{ and } j \neq 0
\end{cases}. \]

- For $n \geq 2$, we have

\[ \psi(u, n|N_1 = k_1) = \sum_{l=u+c+1}^{\infty} f(l, k_1) + \sum_{k_2=0}^{\infty} p_{k_1,k_2} \sum_{j=0}^{u+c} f(j, k_1) \psi(u + \pi - j, n-1|N_1 = k_2). \]
Finite-time ruin probability

- **Example:**
  - Poisson parameter $\lambda = 0.4$; claim amount: $B \sim Geom \left( \frac{1}{3} \right)$, with $E[B] = 2$
  - premium income: 1

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$E[Z_{20}]$</th>
<th>$\text{Var}(Z_{20})$</th>
<th>$\text{VaR}<em>{0.95}(Z</em>{20})$</th>
<th>$\text{TVaR}<em>{0.95}(Z</em>{20})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.2668398</td>
<td>33.264562</td>
<td>12</td>
<td>17.723738</td>
</tr>
<tr>
<td>0.2</td>
<td>4.6316362</td>
<td>40.952906</td>
<td>13</td>
<td>19.654415</td>
</tr>
<tr>
<td>0.5</td>
<td>5.4468555</td>
<td>63.290655</td>
<td>16</td>
<td>24.347508</td>
</tr>
<tr>
<td>0.8</td>
<td>7.0412564</td>
<td>135.708264</td>
<td>22</td>
<td>35.378478</td>
</tr>
</tbody>
</table>

- $E[Z_{20}]$, $\text{Var}(Z_{20})$ and $\text{TVaR}_{0.95}(Z_{20})$ increase as $\alpha \uparrow$ (to be shown)
- **Interpretation:**
  - if the dependence relation is stronger, than the risk process is riskier
  - then the required amount as initial capital has to be higher
Finite-time Gerber-Shiu Function

- Additional assumptions: $\pi \in \mathbb{N}^+, u \in \mathbb{N}$, rv $B$ is defined on $\mathbb{N}$ (to simplify the notation)
- Infinite-time GS function (1998):

$$m_\delta(u) = E \left[ v^\tau w(\tau_-, |U(\tau)|) 1_{\{\tau<\infty\}} |U(0) = u \right]$$

- $w(x, y)$, for $x, y \geq 0$, is the penalty function at the time of ruin for the surplus prior to ruin and the deficit at ruin,
- $v = e^{-\delta} < 1$ is the discounted factor,
- $1_A$ is the indicator function, such that $1_A = 1$, if the event $A$ occurs, and equals 0 otherwise.

Special cases

- when $w(x, y) = 1$, for all $x, y \geq 0$, $m_\delta(u) = \text{Laplace Transform (LT)}$ of $\tau$
- if $\delta = 0$ and $w(x, y) = 1$ for all $x, y \in \mathbb{R}$, $m_\delta(u)$ corresponds to $\psi(u)$
Finite-time Gerber-Shiu Function

- Extended version of the expected discounted penalty function over \( n \) periods:

\[
m_\delta(u, n) = E \left[ \nu^\tau w(U_{\tau^-}, |U(\tau)|) 1_{\{\tau \leq n\}} |U(0) = u\right].
\]  

(5)

(inspired by Morales (2010))

- Special cases

  - when \( w(x, y) = 1 \) for all \( x, y \geq 0 \) and \( \delta = 0 \), \( m_\delta(u, n) = \psi(u, n) \)

- Define

\[
m_\delta(u, n; N_1 = k_1) = E \left[ \nu^\tau w(U_{\tau^-}, |U(\tau)|) 1_{\{\tau \leq n\}} |N_1 = k_1, U(0) = u\right].
\]

We have

\[
m_\delta(u, n) = \sum_{k_1=0}^{\infty} p_{k_1} m_\delta(u, n; N_1 = k_1),
\]

(6)
Proposition. We have

\[ m_\delta(u, n; N_1 = k_1) = \nu \sum_{k_2=0}^{\infty} p_{k_1, k_2} \sum_{j=0}^{u+\pi} f(j; k_1) m_\delta(u + \pi - j, n - 1; N_1 = k_2) \]

\[ + \nu \sum_{j=u+\pi+1}^{\infty} f(j; k_1) w(u, j - u - \pi) \]

where

\[ f(j; k_1) = \begin{cases} 
\Pr(B_1 + \ldots + B_{k_1} = j), & \text{if } k_1 > 0 \\
1, & \text{if } k_1 = 0 \text{ and } j = 0 \\
0, & \text{if } k_1 = 0 \text{ and } j \neq 0 
\end{cases} \]

for \( n \in \mathbb{N}^+ \) and for \( k_1 \in \mathbb{N} \).
Bivariate and trivariate probabilities

- Definitions inspired from Alfa and Drekic (2007) (for $n \in \mathbb{N}^+$)
  - Bivariate joint probabilities for $(\tau, U_{\tau-})$:
    \[
    \zeta_{n,i}(u) = \Pr(\tau = n, U_{\tau-} = i | U(0) = u)
    \]
  - Bivariate joint probabilities for $(\tau, |U_{\tau}|)$:
    \[
    \xi_{n,j}(u) = \Pr(\tau = n, |U_{\tau}| = j | U(0) = u)
    \]
  - Trivariate joint probabilities for $(\tau, U_{\tau-}, |U_{\tau}|)$:
    \[
    \varsigma_{n,i,j}(u) = \Pr(\tau = n, U_{\tau-} = i, |U_{\tau}| = j | U(0) = u)
    \]
    for $i = 0, 1, 2, \ldots; j = 1, 2, \ldots$
We define

$$\zeta_{n,j}^{(k_1)}(u) = \Pr (\tau = n, |U_{\tau}| = j | U(0) = u, N_1 = k_1)$$

for \( n \in \mathbb{N}^+, u \in \mathbb{N}, j \in \mathbb{N} \)

We have the following relationship

$$\zeta_{n,j}(u) = \sum_{k_1=0}^{\infty} p_{k_1} \zeta_{n,j}^{(k_1)}(u).$$
Proposition. Bivariate joint probabilities for \((\tau, |U_\tau|)\).

For \(n = 1\), we have

\[
\xi^{(k_1)}_{1,j}(u) = f(u + c + j; k_1).
\]

where

\[
f(j; k_1) = \begin{cases} 
\Pr (B_1 + \ldots + B_{k_1} = j), & \text{if } k_1 > 0 \\
1, & \text{if } k_1 = 0 \text{ and } j = 0 \\
0, & \text{if } k_1 = 0 \text{ and } j \neq 0 
\end{cases}.
\]

For \(n \geq 2\), we have

\[
\xi^{(k_1)}_{n,j}(u) = \sum_{k_2=0}^{\infty} \Pr (N_2 = k_2|N_1 = k_1) \sum_{l=0}^{u+c} f(l, k_1) \xi^{(k_2)}_{n-1,j}(u + c - l).
\]
Bivariate joint probabilities for time at ruin and deficit at ruin

- **Relations.**
  - Ruin probabilities:
    \[
    \psi(u, n) = \sum_{k=1}^{n} \sum_{j=1}^{\infty} \xi_{k,j}(u).
    \]
  - Moments:
    \[
    E \left[ |U_\tau|^m | \tau \leq n, U(0) = u \right] = \frac{\sum_{k=1}^{n} \sum_{j=1}^{\infty} j^m \xi_{k,j}(u)}{\psi(u, n)}.
    \]
  - More generally, for \( w(x, y) = \varphi(y) \):
    \[
    m_\delta(u, n) = \sum_{k=1}^{n} \sum_{j=1}^{\infty} v^k \varphi(j) \xi_{k,j}(u).
    \]

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Poisson frais

June 18, 2013
Trivariate joint probabilities for time at ruin, surplus before ruin and deficit at ruin

We define

$$\varrho_{n,i,j}(u) = \Pr(\tau = n, U_{\tau^-} = i, |U_{\tau}| = j | U(0) = u, N_1 = k_1),$$

for $n \in \mathbb{N}^+$, $i = 0, 1, \ldots$, $j = 1, 2, \ldots$

We have the following relationship

$$\varrho_{n,i,j}(u) = \sum_{k_1=0}^{\infty} p_{k_1} \times \varrho_{n,i,j}^{(k_1)}(u).$$
Trivariate joint probabilities for time at ruin, surplus before ruin and deficit at ruin

- **Proposition.** Trivariate joint probabilities for $(\tau, U_{\tau-}, |U_\tau|)$.
  - For $n = 1$, we have
    \[ q_{1,i,j}^{(k_1)}(u) = \delta_{i,u} \times f(u + \pi + j, k_1), \]
    where $\delta_{i,u} = \text{Kronecker delta of } (i, u)$
    \[ \delta_{i,u} = \begin{cases} 1, & \text{if } i = u \\ 0, & \text{otherwise} \end{cases} . \]
  - For $n \geq 2$, we have
    \[ q_{n,i,j}^{(k_1)}(u) = \sum_{k_2=0}^{\infty} p_{k_1,k_2} \sum_{l=0}^{u+c} f(l, k_1) q_{n-1,i,j}^{(k_2)}(u + \pi - l). \]
    where
    \[ f(j; k_1) = \begin{cases} \Pr(B_1 + \ldots + B_{k_1} = j), & \text{if } k_1 > 0 \\ 1, & \text{if } k_1 = 0 \text{ and } j = 0 \\ 0, & \text{if } k_1 = 0 \text{ and } j \neq 0 \end{cases} . \]
Trivariate joint probabilities for time at ruin, surplus before ruin and deficit at ruin

**Advantage:**

- We have all the probabilities to compute any quantities of interest
- General relation:

\[ m_\delta(u, n) = \sum_{k=1}^{n} \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \nu^k w(i, j) q_{k,i,j}(u), \text{ for } u \geq 0 \text{ and } n \in \mathbb{N}^+ \]
Infinite-time ruin measures

- Expected discounted penalty function:

\[ m_\delta(u) = E \left[ v^\tau w(U^-_\tau, |U(\tau)|) 1_{\{\tau < \infty\}} | U(0) = u \right] \]

- For computation purposes, we assume that \( N \) is a discrete-time Markov Chain defined on \( A_m = \{0, 1, 2, \ldots, m\} \).

- We compute \( m_\delta(u) \) with

\[ m_\delta(u) = \sum_{k_1=0}^{m} p_{k_1} \times m_\delta(u; N_1 = k_1), \text{ for } u \in \mathbb{N}, \tag{9} \]

where

\[ m_\delta(u; N_1 = k_1) = E \left[ v^\tau w(U^-_\tau, |U(\tau)|) 1_{\{\tau < \infty\}} | U(0) = u, N_1 = k_1 \right], \tag{10} \]

for \( k_1 \in A_m \).
Infinite-time ruin measures

Recursive relation for $m_\delta(u; N_1 = k_1)$ ($u \in \mathbb{N}$):

$$m_\delta(u; N_1 = k_1) = v \sum_{k_2=0}^{m} p_{k_1, k_2} \sum_{j=0}^{u+c} f(j; k_1) m_\delta(u + \pi - j; N_1 = k_2) \quad + \quad v \gamma(u + \pi; N_1 = k_1), \text{ for } k_1 \in A_m,$$

where

$$\gamma(u; N_1 = k_1) = \sum_{j=u+1}^{\infty} w(u - \pi, j - u) f(j; k_1),$$

and

$$f(j; k_1) = \begin{cases} \Pr (B_1 + \ldots + B_{k_1} = j), & \text{if } k_1 > 0 \\ 1, & \text{if } k_1 = 0 \text{ and } j = 0 \\ 0, & \text{if } k_1 = 0 \text{ and } j \neq 0 \end{cases}.$$
Infinite-time ruin measures

- Two approaches to compute the values $m_\delta(u; N_1 = k_1)$
  - Approach #1: Laplace Transform
  - Approach #2: Recursive relation

- In both cases, we need to find the values of $m_\delta(u; N_1 = k_1)$ for $u \in \{0, 1, \ldots, c - 1\}$ and $k_1 \in A_m$.

- In the case when $N$ is a Poisson INAR(1) process, we approximate $N$ by a Markov chain $N^{(m)}$ defined on $A_m$.

- The $m$–truncated joint probability are given by

  \[
  \Pr(N_1^{(m)} = k_1, N_2^{(m)} = k_2) = \frac{\Pr(N_1^{(m)} = k_1, N_2^{(m)} = k_2)}{\Pr(N_1^{(m)} \leq m, N_2^{(m)} \leq m)}, \quad k_1, k_2 \in A_m.
  \]

- $m$ is fixed such that $\Pr(N_1 \leq m, N_2 \leq m)$ is close to 1.
Examine the impact of dependence of various (finite- and infinite-time) ruin measures.

When the claim amount $B^c$ is a continuous rv,

- approximate $B^c$ by a discrete rv $B$ using discretization technics
- use the recursive formulas

Computation time can be significative.
Here is a brief comment about another important Gerber’s contribution: application of theory of martingale in ruin theory.

We consider continuous-time risk models with dependence (see e.g. Albrecher et Teugels (2006), Boudreault et al. (2007), Cossette et al. (2009)).

We consider an insurance portfolio.

Notations
- Continuous rv $X_k$: amount of the $k$th claim ($k \in \mathbb{N}^+$).
- Continuous rv $W_k$: time elapsed between claim $\#(k-1)$ and claim $\#k$ ($k \in \mathbb{N}^+$).

Illustration

\begin{align*}
0 & \quad X_1 & \quad X_2 & \quad X_3 & \quad X_4 & \quad X_5 \\
0 & \quad W_1 & \quad T_1 & \quad W_2 & \quad T_2 & \quad W_3 & \quad T_3 & \quad W_4 & \quad T_4 & \quad W_5 & \quad T_5
\end{align*}
Epilogue – Numerical evaluation of ruin probabilities based on simulation

- Assumption: \( \{(X_k, W_k), k \in \mathbb{N}^+\} \) forms a sequence of i.i.d. pairs of rvs with \((X_k, W_k) \sim (X, W)\) for \( k \in \mathbb{N}^+ \).

- Joint pdf of \((X, W)\): \(f_{X,W}(x, w)\)

- Joint cdf of \((X, W)\): \(F_{X,W}(x, w)\)

- Joint mgf of \((X, W)\): \(M_{X,W}(x, w)\)

- Surplus process: \( U = \{U(t), t \geq 0\} \) où
  
  - \( U(t) = u + ct - \sum_{k=1}^{N(t)} X_k, \quad t > 0 \)
  - \( U(0) = u = \) initial surplus
  
  - \( c = \) premium rate where \( c = (1 + \eta) \frac{E[X]}{E[W]} \) with \( \eta > 0 \)
  
  - Claim number process \( N = \{N(t), t \geq 0\} \) where
    \[ N(t) = \sum_{k=1}^{\infty} 1\{W_1 + \ldots + W_k \leq t\} \]
Random walk: \( Y = \{ Y_k, k \in \mathbb{N} \} \) where \( Y_0 = 0 \) and
\[
Y_k = Y_{k-1} + (X_k - cW_k), \quad k = 1, 2, \ldots
\]

Infinite-time ruin probability: \( \psi(u) = \Pr(Z > u) \) where \( Z = \max_{k \in \mathbb{N}} \{ Y_k \} \).

Lundberg’s equation: \( E[e^{r(X-cW)}] = M_{X,W}(r,-cr) \)

Adjustment coefficient: \( \rho \) is the unique strictly positive solution of
\[
E[e^{r(X-cW)}] = E[e^{rL}] = M_{X,W}(r,-cr) = 1
\]
where \( L = X - cW = \text{net loss rv} \)
Time of ruin: rv $\tau$

Using theory of martingale, Gerber obtained the following result

**Gerber’s exact expression** for $\psi(u)$:

$$
\psi(u) = \frac{e^{-\rho u}}{E \left[ e^{-\rho U(\tau)} \mid \tau < \infty \right]},
$$

for $u \geq 0$. 

$E$ and $\rho$ are defined in the context of the model.
3 approaches for numerical evaluation of $\psi(u)$ based on simulation:

- #1: Crude Monte Carlo simulation method
- #2: MC simulation method based on Gerber’s exact expression
- #3: Importance sampling based change of measure

Number of simulations: $m$

Approach #1 - Crude MC simulation: approximate $\psi(u)$ by

$$\tilde{\psi}(u; \rho, n, m) = \frac{1}{m} \sum_{j=1}^{m} 1\{\tau(j) < n\}$$
Epilogue – Numerical evaluation of ruin probabilities based on simulation

- Approach #2 - MC simulation based on Gerber’s exact expression

\[
\psi(u) = \frac{e^{-\rho u}}{E[e^{-\rho U(\tau)}|\tau < \infty]}
\]

- Approximate \( E[e^{-\rho U(\tau)}|\tau < \infty] \) by \( \frac{\sum_{j=1}^{m} e^{-\rho U(j)}(\tau(j)) \{\tau(j) < n\}}{\sum_{j=1}^{m} \{\tau(j) < n\}} \)

- We obtain

\[
\tilde{\psi}(u; \rho, n, m) = e^{-\rho u} \frac{\frac{1}{m} \sum_{j=1}^{m} \{\tau(j) < n\}}{\frac{1}{m} \sum_{j=1}^{m} e^{-\rho U(j)}(\tau(j)) \{\tau(j) < n\}}
\]

- It seems to correct the bias in the approximation under approach #1
Approach #3 - Importance sampling

Based on another exact expression

\[ \psi(u) = e^{-\rho u} E(\rho) \left[ e^{\rho U(\tau_u)} \right] \]

for \( u \geq 0 \)

\( E(\rho) \) : expectation defined on a new equivalent probability measure (see e.g. Asmussen & Albrecher (2009)).

Define the rv \( L = X - cW \) with pdf \( f_L \)

Then, under new measure probability measure, we have

\[ f_L^{(\rho)}(x) = \frac{e^{\rho x}}{E[e^{\rho L}]} f_L(x) \]

for \( x \in \mathbb{R} \).

Difficulty : simulate under new probability measure
Example: Distribution of \((X, W)\) is based on fgm copula

\[
C_\alpha (u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2),
\]

and \(X \sim \text{Exp} (\beta), W \sim \text{Exp} (\lambda),\)

Joint pdf of \((X, W)\):

\[
f_{X,W} (x_1, x_2) = \beta e^{-\beta x_1} \lambda e^{-\lambda x_2} \\
+ \theta \beta e^{-\beta x_1} \lambda e^{-\lambda x_2} + \theta 2\beta e^{-2\beta x_1} 2\lambda e^{-2\lambda x_2} \\
- \theta 2\beta e^{-2\beta x_1} \lambda e^{-\lambda x_2} - \theta \beta e^{-\beta x_1} 2\lambda e^{-2\lambda x_2}.
\]
Example: (suite)

Joint pdf of \((X, W)\) under change of measure: 
\[
f_{X,W}^{(\rho)}(x, y) = \ldots
\]
\[
\frac{(1 + \theta)}{E[e^{\rho L}]} \left( \frac{\beta}{\beta - \rho} \right) \left( \frac{\lambda}{\lambda + \rho} \right) (\beta - \rho) e^{-(\beta-\rho)x} (\lambda + \rho) e^{-(\lambda+\rho)y}
\]
\[
- \frac{\theta}{E[e^{\rho L}]} \left( \frac{2\beta}{2\beta - \rho} \right) \left( \frac{\lambda}{\lambda + \rho} \right) (2\beta - \rho) e^{-(2\beta-\rho)x} (\lambda + \rho) e^{-(\lambda+\rho)y}
\]
\[
- \frac{\theta}{E[e^{\rho L}]} \left( \frac{\beta}{\beta - \rho} \right) \left( \frac{2\lambda}{2\lambda + \rho} \right) (\beta - \rho) e^{-(\beta-\rho)x} (2\lambda + \rho) e^{-(2\lambda+\rho)y}
\]
\[
+ \frac{\theta}{E[e^{\rho L}]} \left( \frac{2\beta}{2\beta - \rho} \right) \left( \frac{2\lambda}{2\lambda + \rho} \right) (2\beta - \rho) e^{-(2\beta-\rho)x} (2\lambda + \rho) e^{-(2\lambda+\rho)y}
\]
Example: (suite)
- $\beta = 1$, $\lambda = 1$, premium rate $c = 1.5$
- Dependence parameter $\theta = 0.5$
- $\rho = 0.3788264025$

<table>
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<th>$\psi (u)$ Gerber</th>
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Conclusion #2

- Approach #2 (with Gerber’s formula) is easy to apply.
- It is possible to apply both approaches #2 and #3 for many joint distributions for $(X, W)$
  - Bivariate exponential distributions (Downton-Moran, Raftery, etc.)
  - Bivariate gamma distributions
  - Bivariate mixed exponential
  - Bivariate mixed Erlang
  - Etc.
- It is possible to examine various dependence structure and the impact of dependence on $\psi(u)$
- Using results from Schmidli (2010), we use approach #3 to evaluate G-S function.
- Work in progress ..
- Thank you for your attention!